A SHORT AND EASY COURSE

OF

ALGEBRA,

CHIEFLY DESIGNED FOR THE USE OF THE
JUNIOR CLASSES IN SCHOOLS,

WITH A NUMEROUS COLLECTION OF ORIGINAL

Easy Exercises.

THIRD EDITION. (5,000 COPIES)

BY

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PREFACE.

THE Author deems it necessary to state, that this little book is not made up of selected portions of his Edition of *Wood's Algebra,* but is an entirely new and original work, planned and constructed, with no inconsiderable amount of thought and labour, for the special use of three classes of persons, 1st, The *junior boys* in Schools, who, in the Author's opinion, might devote much of the time, now given to *common arithmetic,* more profitably and pleasantly to *Easy Algebra,* superior, as it confessedly is, both for mental exercise and as an instrument of calculation; 2nd, Those *older Students,* who either have not the time or the will to learn more than the *first rudiments* of mathematical analysis; and, 3rd, The *working men,* of small leisure, but good understanding, who are often found (at least in the manufacturing districts) engaged in researches that would do credit even to persons of greater ability, but yet baffled and perplexed for want of a higher power of computation than common arithmetic can supply.

The Author has carefully examined all the books in use at the present time, which profess to have a similar object. Some begin with *incorrect* Definitions, and lead the Student astray at the very outset. Others are arranged in so unconnected a manner, and so entirely without a plan, that one main element of usefulness is wholly wanting—that which constitutes the glory of Euclid—*consecutive reasoning and deduction.* Others, again, professing to be "*Algebra made Easy,*" are really little more than *Arithmetic made Hard.* And the general result of the Author's examination is, he is not afraid to say, that no *Easy* Algebra has hitherto been published, at least in this country, in which the subject is not either

incorrectly treated, badly arranged, or needlessly debased. At th request of many persons, who feel the want of something better the present attempt is made to supply the deficiency, and t public criticism it is now hopefully committed.

It will be seen that the book is printed with the best type an skill of the Pitt Press, regardless of expense, from the Author' conviction, founded on much experience, that the *bad printing* o Mathematics often leads to *bad writing* on the part of the Student which is the source of much subsequent carelessness and error.

T. L.

MORTON RECTORY, near ALFRETON,
March 1, 1850.

ADVERTISEMENT TO THE THIRD EDITION.

THIS Edition differs in no respect from the last, except in the correction of a few typographical errors; but, by increasing the number of copies printed from 3,000 to 5,000, the Author has been enabled considerably to reduce the price of the book, and thus effectually to meet the wishes of many working-men and underpaid schoolmasters, to whom the expenditure of every shilling is a matter of importance.

This little work, it may be proper to observe, contains *all* the Algebra required for the *ordinary* B.A. degree at Cambridge, and is used as a class-book in the principal lecture-rooms of that University. It has been found also, as intended by the Author, to be peculiarly adapted for the adult classes connected with educational institutions in the manufacturing districts, as well as for *commercial* schools generally. The steady sale of the last Edition (3,000 copies) warrants the Author in believing that he has tolerably well hit the mark which he aimed at.

T. L.

MORTON RECTORY,
May 1, 1856.

CONTENTS.

ALGEBRA.

ALGEBRA is most simply defined as *Universal*, or *General*, *Arithmetic*. It is an extension of the powers of common Arithmetic by the use of *letters* to denote numbers, instead of the figures 1, 2, 3, &c.; and it bears somewhat the same relation to Arithmetic that Steam-Power does to ordinary manual labour—inasmuch as what Arithmetic *can* do *Algebra* will often do more easily, and much which Arithmetic cannot do at all *Algebra* can.

To take a simple example, suppose the following question proposed:—

"What number is that which, upon being increased by 10, becomes 3 times as great as it was before?"

The mere *Arithmetician* would probably proceed by the Rule of *Double Position* thus:

1st. Suppose 20 to be the number,

then, since the number increased by 10 becomes 30,

and 3 times 20 is 60,

the error is 30.

2nd. Suppose 10 to be the number,

then, since the number increased by 10 becomes 20,

and 3 times 10 is 30,

the error is 10.

Hence, by *Rule*, 30 times 10, or 300, diminished by 10 times 20, or 200, leaves 100; and this divided by 20, (the difference of the *Errors*), gives 5 for the number required.

Now let the reader compare this *Arithmetical* working with that by which *Algebra* would enable him to attain the same result—not attempting, of course, at present to *understand* the latter, but simply *observing* the *shortness*, and evident simplicity, of the computation.

The *Algebraist* would proceed thus.

Let x be the number,

then $x + 10 = 3x$, by the question,

$2x = 10$,

$x = 5$, the required number.

By means of these four short lines the proposed question is completely solved, and with such *ease* as to put to shame all attempts to solve such questions by the *Arithmetical* Rule.

In like manner it might easily be shewn, if it were not out of place here, that the solution of other questions lies within the power of *Algebra*, which common *Arithmetic* cannot touch at all. But this the Student may be safely left to gather for himself as he proceeds.

DEFINITIONS, FIRST PRINCIPLES, AND NOTATION.

1. *Quantity*, (from the Latin *quantus*, '*how much*') is a word in common use, answering to the question, 'how much', or 'how many', and therefore expressed by some *number*. Thus a *quantity* of persons is expressed or *measured* by the *number* of them—a *quantity* of cloth by the *number* of yards it contains; and so on.

Hence, to express '*any* quantity', as is required to be done in *Algebra*, we must have something which will express '*any* number'; and for this purpose the letters of the Alphabet are found convenient. Thus, for example, instead of writing or saying 'any quantity multiplied by any other quantity', we merely write or say 'a multiplied by b', where a represents, or stands for, *any number*, and b *any other number**. So that, just as the operations of Arithmetic are simplified and abridged by using the figures 1, 2, 3, &c. instead of the *words*, one, two, three, &c., the operations of Algebra are abridged by using, instead of *words*, the letters a, b, c, &c., x, y, z, to represent, or stand for, *general numbers*.

Various *Signs* or *Symbols* also are used, for the sake of convenience, to express the various *Arithmetical Operations* of Addition, Subtraction, Multiplication, Division, &c. These may be *any* distinct marks which Algebraists can agree upon. At present they are as follow:

2. (ADDITION). $+$ is read *plus* (Latin for *more*), and signifies that the quantity which comes next after it is *to be added* to that which goes before. Thus $a+b$, (which is read *a plus b*), signifies that the quantity represented by b

* To express 'any quantity multiplied by any *other* quantity' it would not be correct to say 'a multiplied by a': this would only express 'any quantity multiplied by *itself*'.

is *to be added* to the quantity represented by a. If a stand for 5, and b for 7, then $a+b$ is $5+7$, and is equal to 12. If also c stand for 4, then $a+b+c$, (which is read *a plus b plus c*), is equal to $12+4$, or 16.

3. (SUBTRACTION). $-$ is read *minus* (Latin for *less*), and signifies that the quantity which comes next after it is *to be subtracted* from that which goes before. Thus $a-b$, (which is read *a minus b*), signifies that the quantity b is *to be subtracted* from the quantity a. If a stand for 10, and b for 6, then $a-b$ is $10-6$, and is equal to 4. If also c stand for 3, then $a-b-c$, (which is read *a minus b minus c*), is equal to $4-3$, or 1.

[*Exercises* A, 1...4, page 5.]

4. (MULTIPLICATION). $\times$ is read *into*, or *times*, and signifies that the quantity which comes next after it is *to be multiplied* by that which goes before. Thus $a\times b$, (which is read *a into b*, or *a times b*), signifies that b is *to be multiplied* by a, or taken *a times*. If a stand for 6, and b for 4, then $a\times b$ is 6 *times* 4, and is equal to 24. If also c stand for 2, then $a\times b\times c$, (which is read *a into b into c*), is equal to 24×2, or 48.

Similarly $3\times x$ means x taken 3 *times*, and is read '3 *times x*', or more usually '*three x*', meaning '*three x*'s'.

This symbol $\times$ is often abbreviated to a *point*, or even omitted altogether, where it must be *understood*. Thus $a\times b$, $a.b$, and ab, all mean the same thing, viz. *a times b*. Similarly $3x$ will stand for '3 *times x*'; '$7y$ for 7 *times y*'; and so on.

Again $a\times b\times c$, $a.b.c$, and abc, all mean the same thing; and $3xy$ means '3 *times* the product of x and y'.

Observe, then, that when *no sign* of operation is found between two letters standing together, or between a figure and a letter, as in ab, $3x$, the word '*times*' must be *understood* between them, as *a times b*, 3 *times x*. We may omit the word in *reading* Algebra, but it is always to be *understood*. Care must be taken by the learner not to confound $3x$ with $3+x$, that is, 3 *times x* with 3 *plus x*; and so also in other like cases. There is the more need to be careful here, because in *Common* Arithmetic the case is precisely *reversed*. *There* the sign of *Addition* is constantly omitted and understood: for instance, $2\frac{1}{2}$ stands for $2+\frac{1}{2}$; 23 means $20+3$: and so on. Hence although the *Sign of Multiplication* may be omitted between two *letters*, or *a figure and a letter*, it is

obvious that it must never be omitted between two *Arithmetical Numbers* which are to be multiplied together: for instance 57 cannot be used conveniently to stand for 5×7. Also in such cases it is not well to use the *abbreviated* sign, as 5.7, on account of its similarity to the *decimal point*: but between numerals, which are to be multiplied together, the full sign $\times$ should always be used.

[*Exercises* A, 5...12, page 5.]

5. Again, since we know that 3×4 is equal to 4×3,

5×7 .... 7×5,

6×10 .... 10×6,

and so on, whatever numbers we take; therefore we may say generally, that

$a\times b$ is equal to $b\times a$,

or ab is equal to ba.

6. (FACTORS). Every quantity which enters *as a multiplier* to make up a *product* is called a *factor* of that product. Thus 5 and 7 are the *factors* of 35, because 5×7 *makes* 35; 3 and x are the *factors* of $3x$; a and b are the *factors* of ab; and so on.

Observe, it must be either an actual *product*, as 35, or the equivalent expression, as 5×7, which has *factors*: in other words, the existence of *factors* presupposes a *multiplication* either already effected, or to be effected. So that any quantity which has not been, and cannot be, *made or produced by multiplication*, has no *factors*. Thus each of the quantities 7, 13, 17, has no *factors*, since there is no number except 1 which by multiplication will produce any of them. If, however, 1 be considered as a number, then the *factors* in each of these cases are respectively 1 and 7, 1 and 13, 1 and 17.

7. (COEFFICIENTS). In the quantity ab, or its equal ba, a is the *co-factor* of b, and b is the *co-factor* of a; (just as we say, of two persons in partnership, that *each* is the *co-partner* of the *other*). But instead of '*co-factor*' the word '*coefficient*' is generally used; so that, in ab, a is called the *coefficient* of b, and b the *coefficient* of a. Thus, in $3x$, the *coefficient* of x is 3, because 3 is the *co-factor* of x to make $3x$.

Also, in $3xy$, 3 is the *co-factor* or *coefficient* of xy; $3x$ is the *co-factor* or *coefficient* of y; $3y$ is the *co-factor* or *coefficient* of x.

In $2abc$, $2ab$ is the *coefficient* of c; $2ac$ is the *coefficient* of b; $2bc$ is the *coefficient* of a; 2 is the *coefficient* of abc; $2a$ is the *coefficient* of bc; and so on.

In a, the only *factors* being 1 and a, the *co-factor* or *coefficient* of a is 1.

In other words, *the number of times* a quantity is taken is the *coefficient* of that quantity. Taking the same examples, $3x$ is *3 times* x, or 3 is the *coefficient* of x; $3xy$ is *3 times* xy, or 3 is the *coefficient* of xy; or it is $3x$ *times* y, that is, $3x$ is the *coefficient* of y; or it is $3y$ *times* x, (since xy is equal to yx) (Art. 5)*, that is, $3y$ is the *coefficient* of x. In a, a is taken 1 *time* or once, and 1 is the *coefficient* of a.

There is no impropriety in making use of such an expression as $3x$ *times*, or $2ab$ *times*, because each letter represents a *number*. (Art. 1). Thus, in $3xy$, if x stand for 10, $3x$ would be 30, and $3x$ *times* y would be $30y$.

[*Exercises* A, 13...20, page 6.]

8. (DIVISION). $\div$ is read '*divided by*', or more shortly '*by*', and signifies that the quantity which comes next after it is to be the *divisor* of that which goes before. Thus $a \div b$ (which is read a *by* b) signifies that a is to be *divided by* b. Thus $8 \div 4$ is equal to 2. But this symbol for division is not much used, because the *fraction* $\frac{a}{b}$, (which is also read a *by* b), means the same thing as $a \div b$, and is found more convenient: thus $\frac{8}{4}$ is the same as $8 \div 4$, both being equal to 2.

[*Exercises* A, 21...30, page 6.]

EXERCISES. A.

If a stand for 10, b for 3, and x for 7, what is the value of each of the following quantities†?

(1) $a + b + x$.

(2) $a + b - x$.

(3) $a - b + x$.

(4) $a - b - x$.

(5) $2a - x$.

(6) $4a + 3b - 2x$.

(7) $7a + 2b - 2x$.

(8) $5a - 4b - 4x$.

(9) $2ab - 3x$.

(10) $2a + 5 - 3bx + 100$.

(11) $7ab - abx$.

(12) $3a + bx - xx$.

* This is the usual way of referring the reader to a previous clause or *article*.

† The answers to all the Exercises will be found at the end of the book.

(13) What is the *coefficient* of x in $3ax$?

(14) What is the *coefficient* of x in $6abx$?

(15) What is the *coefficient* of bx in $6abx$?

(16) What is the *coefficient* of a in each of the quantities $2a$, $2ab$, abx, $3abx$, ma, axx, pax, $abxy$?

(17) What is the *coefficient* of 25 in 125?

(18) What is the difference between $3+x$, and $3x$, when x stands for 7?

(19) What is the difference between $3a+x$, and $3a-x$, when a stands for 10, and x for 6?

(20) What is the difference between $3a+x$, and $3ax$, when a stands for 3, and x for 2?

Find the value of each of the following quantities, when a stands for 10, b for 3, and x for 7:

(21) $3ax \div 7$.

(22) $3ax \div 7b$.

(23) $\dfrac{2a+x}{b}$.

(24) $\dfrac{3b+3x}{a}$.

(25) $\dfrac{a-x}{b}$.

(26) $\dfrac{3a}{b}+2x-\dfrac{abx}{21a}$.

(27) $\dfrac{5a+x}{b}-\dfrac{5b+a}{2x-3b}$.

(28) $\dfrac{3x}{4a+2}+\dfrac{4bx}{10a-16}$.

(29) $\dfrac{2a+4b}{3x-a-b}-\dfrac{a-2b}{x-b}$.

(30) $\dfrac{ma}{b+x}+\dfrac{nb}{a-x}-\dfrac{px}{a-b}$.

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9. (Involution). If a quantity is multiplied by *itself* any number of times, the quantity is said to be *involved*, and the operation is called *Involution*. Here the following convenient *abbreviations* are used:—

$a \times a$ we write a^2, which is called the 2nd *power* of a;

$a \times a \times a$ .... a^3, ... 3rd ....

$a \times a \times a \times a$ .... a^4, ... 4th ....

and so on; a, or a^1, being called the 1st power of a.

Also a^2 is called the *square* of a, and is read 'a square';
a^3 .... *cube* .... 'a cube';
a^4 is read 'a to the 4th'; a^5 is read 'a to the 5th'; and so on.

Observe, a is the same as a^1, not a^0.

The small figures ^{1}, ^{2}, ^{3}, ^{4}, &c. placed as above to the right of quantities are called their *indices*, because they *point out* the *power* of the quantities.

So, then, we have now an abbreviated form for both $a+a$, and $a\times a$. The former is written $2a$, the latter a^2.

If a stand for 4, $2a$ is equal to 8, and a^2 is equal to 16. Also, it must be carefully remembered that $2a^2$ does not mean the square of $2a$, but twice the square of a.

[*Exercises* B, 1...8, page 8.]

10. (EVOLUTION). *Evolution* is exactly the reverse operation to *Involution*. It is the process by which we '*evolve*' or '*extract*' the original quantity, called '*root*', by the *Involution* of which a proposed quantity is produced. For example, the '*square root*' of any proposed quantity is that quantity which, being multiplied by itself, or *squared*, will produce the proposed quantity. Also the '*cube root*' is that quantity which being *cubed* will produce the proposed quantity. Thus 3 is the *square root* of 9, because 3 *squared*, or 3×3, is equal to 9; and 3 is the *cube root* of 27, because 3 *cubed*, or $3\times3\times3$, is equal to 27.

Again, a is the square root of a^2, because $a\times a$ gives a^2; also a is the cube root of a^3, because $a\times a\times a$ gives a^3.

The abbreviations here are these:

Instead of writing 'the square root of' we write $\sqrt[2]{}$, or $\sqrt{}$;
.... 'the cube root of' .... $\sqrt[3]{}$.

The symbol $\sqrt{}$ is a corruption of the letter r, the first letter of the word '*root*', and as the letter r is now often used in Algebra for other purposes, the more unlike $\sqrt{}$ is made to its original form the better.

For the *square root* $\sqrt{}$ is commonly used, not $\sqrt[2]{}$, which is more strictly correct. And it is read simply '*root*', but meaning the *square* root. Thus $\sqrt{a}$ is read 'root a', meaning the *square* root of a.

Again, just as $a+a$ is written $2a$, so $\sqrt{a}+\sqrt{a}$, or twice the square root of a, is written $2\sqrt{a}$, and read 'twice root a'.

Also $\sqrt{ab}$ signifies 'the square root of a *times* b';

$\sqrt{a+b}$ .... 'the square root of a *plus* b', that is of the *sum* of a and b; and so on, the symbol $\sqrt{}$ being extended in its upper limb to cover the whole quantity of which the root is to be taken.

Hence, if a stand for 16, and b for 9, $\sqrt{a+b}$ is equal to $\sqrt{25}$, or 5; and $\sqrt{ab}$ is equal to $\sqrt{144}$, or 12.

Also $\sqrt{\dfrac{a}{b}}$ means that 'the square root of the *fraction* $\dfrac{a}{b}$, is to be taken; but $\dfrac{\sqrt{a}}{b}$ means that 'the square root of a is to be divided by b'.

[*Exercises* B, 9...21.]

EXERCISES. B.

If a stand for 1, b for 9, and c for 8, find the value of each of the following quantities:

(1) $a^2+b^2-c^2$.

(2) $13a^2+3b^2-4c^2$.

(3) $5abc-22b^2+3c^3$.

(4) a^2b+b^2c.

(5) $12ab^2+20a^2b-2bc^2$.

(6) $\dfrac{b^2}{a}+\dfrac{a^2}{b}-\dfrac{c^2}{2a}$.

(7) $\dfrac{8ab^2}{3c}-\dfrac{9ac^2}{2bc}$.

(8) $ma^2+nb^2-pc^2$.

(9) $2\sqrt{b}-\sqrt{2c}$.

(10) $\sqrt{ab}+\sqrt{a^2b}$.

(11) $a+\sqrt{b}-\sqrt{ab}+2\sqrt{2bc}$.

(12) $\sqrt{2c+b}-\sqrt{2b-2a}$.

(13) $m\sqrt{\dfrac{b}{a}}+n\sqrt{\dfrac{bc}{2}}-p\sqrt{2ac}$.

(14) $\sqrt[3]{ac}+\sqrt{4b}-2\sqrt[3]{c}$.

(15) $\sqrt{b+c-a}-\sqrt[3]{3b-2c-3a}$.

(16) $\sqrt[3]{a}+\sqrt{2c}\,.\sqrt[3]{\dfrac{bc}{9}}-4\sqrt[3]{b-a}$.

(17) What is the difference between $3a$, and a^3, when a stands for 2?

(18) What is the difference between $2\sqrt{x}$, and $2+\sqrt{x}$, when x is 100?

(19) What is the difference between $3\sqrt{x}$, and $\sqrt[3]{x}$, when x is 64?

(20) What is the difference between $\sqrt{a+b}$, and $\sqrt{a}+b$, when a stands for 1, and b for 8?

(21) What is the difference between $\sqrt{\frac{a}{b}}$, and $\frac{\sqrt{a}}{b}$, when a stands for 16, and b for 4?

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11. The following are certain other symbols, or abbreviations, in common use:—

$=$ stands for '*is equal to*', and is read '*equals*'. Thus $2+4=6$; $a+x=b$, is read, 'a plus x equals b', and means that the sum of a and x is equal to b; $8 \div 4 = 2$; $\sqrt{25}=5$; and so on.

$>$ stands for '*is greater than*'; thus $a > b$ signifies that a is *greater than* b.

$<$ stands for '*is less than*'; thus $a < b$ signifies that a is *less than* b.

$\therefore$ stands for '*therefore*'; $\because$ for '*since*', or '*because*'.

12. (Terms). Algebraical quantities are said to consist of one or more '*terms*', according as they are composed of one or more parts separated by the signs + or −. Thus a is a quantity of one *term*: so also is each of the quantities $2a$, ab, a^2b, abc, and so on. Again, $a+b$ is a quantity of two *terms*: so also is $a-b$, and $ab+ac$, and a^2b-abc, and so on. A quantity of three *terms* is of the form $a+b+c$; &c.

13. (Positive and Negative Quantities). Any quantity *of one term* preceded by the sign +, taken together with the sign, is called *a positive quantity*. Any quantity of one term preceded by the sign −, taken together with the sign, is called *a negative quantity*. And since $+a$ is the same as a, (for it signifies *a to be added to* 0) all quantities of one term, without either + or − preceding, are *positive quantities*. Any quantity *of more terms than one* will be *positive* or *negative* according as the sum of the *positive terms* taken together exceeds or falls short of the sum of the *negative terms* taken together.

This may be illustrated by the case of a person taking an account of what money he is worth. He first puts down his stock on hand, which may be represented by *a, without sign:* then the amount of the sums due to him from others, which may be represented by *b, with a + sign before it, because it is to be added to a.* Then his debts are to be *subtracted,* and their amount may be represented by $-c$, a

negative quantity; so that the money which he is really worth will be represented by $a+b-c$.

In a case like this a person easily distinguishes between *positive and negative quantities;* and if he finds that c is greater than $a+b$, he has no difficulty in fully comprehending the *meaning* of *a negative quantity.*

N.B. Although there are many '*signs*' in Algebra, as the preceding pages testify, yet when we speak simply of '*the* sign' *of a quantity*, we always mean either + or −, that is, simply to express whether the quantity is *positive* or *negative.*

QUESTIONS.

1. How do you define Algebra, and of what use is it?

2. What do you mean by '*Quantity*'?

3. Why do we use *letters* to represent *quantities* in Algebra?

4. What do you mean by $a+b$? Write it at full length in *words*. Does $2+5$ mean that 2 is *added* to 5?

5. What does 23 mean in *Arithmetic*? What does ab mean in *Algebra*?

6. What is $3a$ an abridgment of? Which is greater $3a$, or $3a-b$?

7. If a stand for 1, b for 2, and c for 3, would abc be equal to 123? If not, what is it equal to?

8. What does $5\frac{2}{3}$ mean in *Arithmetic*? What does $a\frac{b}{c}$ mean in *Algebra*?

9. According to the definition of +, what is the meaning of $+x$ standing thus alone?

10. Is the quantity, whose *factors* are 6 and 7, the same as that whose *factors* are 7 and 6? What is the quantity? Is ab a *factor* of abc? What is *understood* between any two contiguous letters in abc?

11. What is signified by $ab-c$? Write it in *words*.

12. What is signified by $2ab+3$? Write it in *words*.

ADDITION.

14. DEFINITION. *Like* quantities are such as differ only in the *numerical coefficients.*

Thus $4a$, $7a$, $10a$, are *like* quantities; so also are $3ab$, $6ab$, ab: so again are a^2, $3a^2$, $5a^2$; and so on.

DEF. *Unlike* quantities are such as are either represented by different letters, or by different combinations of the same letters.

Thus a, b, x are *unlike* quantities; so also are $2a$, $3b$, $4x$; so again are ab, a^2b, a^2b^2; and so on.

Ex. 1. Group together *like* quantities, with their proper signs, from $5a-3b$, $4a+7b$, and $-8a-5b$.

Ans.		
	$+5a$	$-3b$
	$+4a$	$+7b$
	$-8a$	$-5b$

Here the quantities in each column are *like*, but the two columns are *unlike.*

Ex. 2. Group together *like* quantities, with their proper signs, from

$$a^3+3a^2b+3ab^2+2a^3+2b^3+5ab^2-8ac^2-a^2b-b^3.$$

Ans.					
	$+a^3$	$+3a^2b$	$+3ab^2$	$-8ac^2$	$+2b^3$
	$+2a^3$	$-a^2b$	$+5ab^2$		$-b^3$

Ex. 3. Group together *like* quantities, with their proper signs, from $2a-3b+7bc+b^2c-5abc+2xy-3x^2+5b^2+7b^2c-9a-2b^2+6b+10a-5x^2-xy+x^2+abc-2bc+c^2-b-3c^2$.

Ans.									
	$+2a$	$-3b$	$+7bc$	$+b^2c$	$-5abc$	$+2xy$	$-3x^2$	$+5b^2$	$+c^2$
	$-9a$	$+6b$	$-2bc$	$+7b^2c$	$+abc$	$-xy$	$-5x^2$	$-2b^2$	$-3c^2$
	$+10a$	$-b$					$+x^2$		

15. *To add* LIKE *quantities together.*

RULE 1st. *When the quantities to be added together are preceded by the same sign, either + or −, {bearing in mind that for such as have no sign + is to be understood* (Art. 13)} *the addition is performed by taking the sum of all the numerical coefficients, with that sign, for the new coefficient, and annexing to the right hand of it the common letter or letters.*

Thus $5a$ and $4a$ added together make $9a$; for $5a$ means 5 *times* a, that is, $a+a+a+a+a$, and $4a$ means 4 *times* a, that is, $a+a+a+a$, therefore $5a$ added to $4a$ is clearly a taken 9 *times* or $9a$. Again, $-2b$ means $2b$ *to be subtracted,*

and $-3b$ means $3b$ *to be subtracted*, therefore $-2b$ added to $-3b$ is $2b$ to be subtracted added to $3b$ to be subtracted, which is clearly $5b$ to be subtracted, that is, $-5b$. And the same reasoning will apply to *any like quantities.*

RULE 2nd. *When the like quantities to be added together have different signs, some +, others −, the addition is performed by taking the difference between the sum of the positive, and the sum of the negative, coefficients, with the sign of the greater sum for the new coefficient, and annexing the common letter or letters.*

Thus if $5a$, or $+5a$, is to be added to $-2a$, this can only mean that we are to find the *joint* effect of adding $5a$ and *subtracting* $2a$, which clearly leaves $3a$ *to be added,* that is, $+3a$.

Again, to add together $3a$, $-2a$, $-5a$, and $10a$; here we have $13a$ *positive,* and $7a$ *negative;* therefore upon the whole we have $6a$ *positive,* that is, the sum of the quantities is $+6a$.

Also, to add together $-3a$, $2a$, $5a$, and $-10a$: here we have $7a$ positive, and $13a$ negative, therefore upon the whole we have $6a$ *negative,* that is, the sum of the quantities is $-6a$.

The following additions are correctly performed for the learner's inspection:—

	$2x$	$3ab$	$-5a$	$-ab$
	$4x$	$5ab$	$-6a$	$-5ab$
	$7x$	$2ab$	$-2a$	$-3ab$
	x	ab	$-a$	$-2ab$
Sum =	$14x$	$11ab$	$-14a$	$-11ab$

	$4a$	$2xy$	$3a^2$	$15ab^2$
	$-7a$	$7xy$	$2a^2$	$-7ab^2$
	$5a$	$-6xy$	$-6a^2$	$-4ab^2$
	$-a$	$-xy$	$7a^2$	$9ab^2$
	a	$+5xy$	$-4a^2$	$-3ab^2$
	$10a$	xy	$-5a^2$	$-ab^2$
	$-6a$	$-8xy$	$10a^2$	$-10ab^2$
Sum =	$6a$	0	$7a^2$	$-ab^2$

RULE 3rd. *When quantities of two or more terms are to be added together, like terms may be added separately, and these sums, with their proper signs, placed in one line, will be the sum required.*

Thus if $2a+3b$ is to be added to $3a+4b$, $2a$ and $3a$ being added together make $5a$; and $+3b$ added to $+4b$ makes $+7b$; so that the whole sum required is $5a+7b$. Or, if $3a-4b$ is to be added to $2a+3b$, $2a$ and $3a$ make $5a$, $-4b$ and $+3b$ leaves $-b$; so that the sum required is $5a-b$.

In fact, since $2a+3b$ means no more than that $3b$ is *to be added* to $2a$, and $3a+4b$ that $4b$ is *to be added* to $3a$, when we say, add together $2a+3b$, and $3a+4b$, it is plainly the same as saying, add together $2a$, $3b$, $3a$, and $4b$.

And, indeed, this is no more than is done in Arithmetic, when we add two or more sums of money together, we separate like terms from all and add them together, all the pence together, and all the shillings together, and all the pounds together.

Ex. 1. Find the sum of $5a-3b$, and $4a-7b$.

$$\begin{array}{rr} & 5a-3b \\ & 4a-7b \\ \hline \text{Sum} = & 9a-10b \end{array}$$

Here $5a$ added to $4a$ is $9a$; and $3b$ to be subtracted together with $7b$ to be subtracted is manifestly $10b$ to be subtracted, that is, $-10b$.

Ex. 2. Find the sum of $5a-3b$, and $4a+7b$.

$$\begin{array}{rr} & 5a-3b \\ & 4a+7b \\ \hline \text{Sum} = & 9a+4b \end{array}$$

Here $5a$ added to $4a$ is $9a$; $7b$ to be added and $3b$ to be subtracted leaves $4b$ to be added, that is, $+4b$.

Ex. 3. Find the sum of $5a-3b$, $4a+7b$, and $-8a-5b$.

$$\begin{array}{rr} & 5a-3b \\ & 4a+7b \\ & -8a-5b \\ \hline \text{Sum} = & a-b \end{array}$$

Here we have $9a$ positive, and $8a$ negative, which leaves $1a$, or a, positive: then we have $7b$ positive, and $8b$ negative, which leaves $1b$, or b negative.

Ex. 4. Find the sum of $3a^2+4bc-e^2+10$, $-5a^2+6bc+2e^2-15$, and $-4a^2-9bc-10e^2+21$.

Grouping *like* quantities in order under each other, we have

$$\begin{array}{r} 3a^2+4bc-\ \ e^2+10 \\ -5a^2+6bc+\ 2e^2-15 \\ -4a^2-9bc-10e^2+21 \\ \hline \text{Sum} = -6a^2+\ bc-\ 9e^2+16 \\ \hline \end{array}$$

The first column of *like* quantities consists of $3a^2$ positive, and $9a^2$ negative, which leaves $6a^2$ negative, or $-6a^2$. The second is $10bc$ positive, and $9bc$ negative, which leaves $1bc$, or bc, positive, or $+bc$. The next is $2e^2$ positive, and $11e^2$ negative, which leaves $9e^2$ negative, or $-9e^2$. The last is 31 positive, and 15 negative, which leaves 16 positive; or +16.

[*Exercises* C, 1...24, page 15.]

16. *To add* UNLIKE *quantities together.*

RULE. *Strictly speaking this is impossible. All that is meant is, to combine the quantities together in a more convenient form with the necessary algebraical signs.*

Thus, in this sense, the *sum* of a, $-b$, c, $-d$, and e, is $a-b+c-d+e$. The quantities are in no sense actually added together; but they are so placed as to express *algebraically* the aggregate of them. For it must be borne in mind that $a+b$ does not signify that b *is* added to a, but that it *is to be* added, when we know the numbers which a and b stand for.

17. RULE. *If the quantities to be added together consist of both like and unlike terms, the like terms must be added by the method of Art.* 15, *and the unlike affixed to that sum in the same line with their proper signs.*

It is immaterial in what order the quantities are set down in the sum, provided each has its proper sign. But it is usual to keep the order of the alphabet, unless there be some special reason for a different arrangement.

Ex. 1. Add together $a+2b-c$, $a-5e+2c$, and $x+y+3e$.

Here a and a are *like*,	$a+2b-c$
$-5e$ and $+3e$	$a-5e+2c$
$-c$ and $+2c$	$3e+x+y$
the rest are *unlike*.	Sum $=2a+2b+c-2e+x+y$

Ex. 2. Add together $3a^2-bc$, $2b^2-ac$, $4c^2-ab$, and $a^2+b^2-c^2$.

Here $3a^2$ and a^2 are *like*,
$2b^2$ and $+b^2$ ......
$4c^2$ and $-c^2$ ......
the rest are *unlike*.

$$\begin{array}{l} 3a^2 - bc \\ \quad\quad 2b^2 - ac \\ \quad\quad\quad\quad 4c^2 - ab \\ a^2 + b^2 - c^2 \\ \hline \text{Sum} = 4a^2 + 3b^2 + 3c^2 - ab - ac - bc \end{array}$$

Ex. 3. Add together $xy-1$, x^2+2, and y^2+3.

Here the terms are all *unlike*, except -1, $+2$, and $+3$.

$$\begin{array}{l} xy - 1 \\ x^2 + 2 \\ y^2 + 3 \\ \hline \text{Sum} = x^2 + xy + y^2 + 4 \end{array}$$

[*Exercises* C, 25...30.]

18. The Rules above given for the Addition of *like* and *unlike* algebraical quantities are in no wise different from those employed in *Arithmetic*. For suppose we have to add together 3 hundreds, and 4 hundreds, we combine these *like quantities* by taking the sum of the *coefficients* 3 and 4, so as to make 7 hundreds. But if we have to add together 3 hundreds, 5 tens, and 6 units, these, being *unlike quantities*, cannot be added in the same sense, but are merely collected together in one line, 3 hundreds + 5 tens + 6 units, which for convenience is written shortly 356.

EXERCISES. C.

Add together

(1) $a+b$, and $a+b$.

(2) $a+b$, and $a-b$.

(3) $a-b$, and $a-b$.

(4) $a-b+c$, and $a+b-c$.

(5) $a-b+c$, and $a+b+c$.

(6) $1-2m+3n$, and $3m-2n+1$.

(7) $5m+3$, and $2m-4$.

(8) $3xy-2x$, and $xy+6x$.

(9) $4p-2q+1$, and $7-3p+q$.

(10) $5ab-2bc$, and $ab+bc$.

(11) $2ax+3by$, and $ax-by$.

(12) $3a-2b+4c$, and $2a-3b+c$.

(13) $xy+x-7$, and $3xy-2x+3$.

(14) $p+q-pq$, and $2pq-3p+2q$.

(15) $p^2+2pq+q^2$, and $p^2-2pq+q^2$.

(16) $7ab - 5ac + 1$, and $ab + 6ac - 2$.

(17) $7x - 6y$, $-x - 3y$, $-x + y$, $-2x + 3y$, and $x + 8y$.

(18) $3 - a$, $-8 - a$, $7a - 1$, $-a - 1$, and $9 + a$.

(19) $a - 3b + 3c - d$, and $a + 3b + 3c + d$.

(20) $a^2 + 2ab + b^2$, and $2a^2 - ab - 3b^2$.

(21) $3x^2 - 6x + 5$, $2x - 3 - x^2$, and $4 - x - 2x^2$.

(22) $ac + bd$, $bd - cd$, and $ac + cd$.

(23) $ax - by$, $x + y$, and $ax - x - by - y$.

(24) $4x^2y - 4axy - 2a^2x + 2x^3$, and $x^2y + axy + a^2x - x^3$.

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(25) $8mn + m$, and $1 - n - 7mn$.

(26) $9x - 8y - 7$, and $3z - 9x + 6y + 7$.

(27) $x^3 - 2ax^2 + a^2x$, $x^3 + 3ax^2$, and $2a^3 - ax^2 - a^2x$.

(28) $a^2 - 3ab - \frac{2}{3}b^2$, $2b^2 - \frac{2}{3}b^3 + c^2$, $ab - \frac{1}{3}b^2 + b^3$, and $2ab - \frac{1}{3}b^3$.

(29) $\frac{1}{4}x^2 + 2xy$, $\frac{3}{4}x^2 - xy + y^2$, and $mx + ny$.

(30) $ad + 2bd - 3cd$, $\frac{1}{2}ad - \frac{1}{2}bd$, and $\frac{1}{2}ab + 2cd - ac$.

QUESTIONS.

1. Are $4a$, and $4b$, '*like*' or '*unlike*' quantities?
2. Are $4a$, and $-3a$, '*like*' or '*unlike*' quantities?
3. Are x^2, and x^3, '*like*' or '*unlike*' quantities?
4. Are $2xy$, and $\frac{1}{2}xy$, '*like*' or '*unlike*' quantities?
5. Are $5x^2y$, and $4yx^2$, '*like*' or '*unlike*' quantities?
6. Express the '*sum*' of each pair of quantities in each of the preceding questions.

SUBTRACTION.

19. *To* SUBTRACT, *or take away, one quantity from another.*

RULE. *Change the sign of the quantity to be subtracted, + into −, or − into +, as the case may be, and then* ADD *the quantities together by the rules for Addition.*

1st. If the quantities are *like*, and of the same sign, that is, both positive or both negative, their *difference* is found by taking the *difference* of the *numerical coefficients*, with that sign, for the new coefficient, and annexing to the right hand the common letter or letters.

Thus, suppose $2a$ is to be taken from $5a$, since $5a = 3a + 2a$, $\therefore$ $2a$, which is the same as $+2a$, taken from $5a$, leaves $3a$.

Or, suppose $-2a$ is to be taken from $-5a$, since $-5a = -3a - 2a$, $\therefore$ $-2a$ taken from $-5a$ leaves $-3a$.

2nd. If the quantities are *like*, but of different signs, that is, one *positive*, and the other *negative*, their *difference* is found by taking the *sum* of the numerical coefficients for the new coefficient, with the sign of the quantity from which the other is to be taken, and annexing the common letter or letters.

For, suppose $2a$ is to be taken from $-5a$, the result is obviously *equal to* $-5a-2a$, which we know to be $-7a$. Or, if $-2a$ is to be taken from $5a$, since $5a = 7a - 2a$, $\therefore$ $-2a$ taken away from $5a$ leaves $7a$.

3rd. If the quantities are *unlike*, their difference cannot be found, but can only be *expressed* by writing the quantities in one line with the proper signs.

Thus, if b is to be taken from a, this is expressed by $a - b$. If $-b$ is to be taken from a, since $a = a + b - b$, $\therefore$ $-b$ taken from a leaves $a + b$.

Hence, collecting together the several cases which can occur,

$2a$, or $+2a$ taken from $5a$, or $+5a$, leaves $+3a$,
$-2a$ .... $-5a$, ... $-3a$,
$2a$, or $+2a$ .... $-5a$, ... $-7a$,
$-2a$ .... $+5a$, ... $+7a$;
b, or $+b$ .... a, or $+a$, ... $+a-b$,
$-b$ .... a, or $+a$, ... $+a+b$;

and observing that the same reasoning will apply to any other quantities besides those here used, it appears that we embrace all cases by the Rule above stated.

EXAMPLES.

1. From $3a$ take a Diff. $= 2a$	2. From $7a$ take $6a$ a	3. From a take a 0
4. From $3a$ take $-a$ Diff. $= 4a$	5. From $7a$ take $-6a$ $13a$	6. From a take $-a$ $2a$

7. From $-3a$ take a Diff. $=-4a$

8. From $-7a$ take $6a$; $-13a$

9. From $-a$ take a; $-2a$

10. From $-3a$ take $-a$ Diff. $=-2a$

11. From $-7a$ take $-6a$; $-a$

12. From $-a$ take $-a$; 0

13. From $a+b$ take $a-b$ Diff. $=2b$

14. From $a-b$ take $a+b$; $-2b$

15. From $y+ax$ take $y-ax$; $2ax$

16. From $3a-4b+6c$ take $a-2b+9c$ Diff. $=2a-2b-3c$

17. From $7a-2b+4c-2$ take $6a-6b+4c-1$; $a+4b-1$

18. From $2a-6ab-ac+5$ take $5a-8ab-2ac-1$ Diff. $=-3a+2ab+ac+6$

19. From $3xy-x^2-y^2+a$ take $2xy+x^2+2y^2-b$; $xy-2x^2-3y^2+a+b$

20. From $a^2+2ab-3c^2$ take $2a^2-5ab-7c^2$ Diff. $=-a^2+7ab+4c^2$

21. From $5x^2-xy+y^2$ take $-x^2+4xy+3y^2$; $6x^2-5xy-2y^2$

22. From $8a^2+x^2-5b^2-5c^2$ take $x^2+2b^2-5c^2$ Diff. $=8a^2-7b^2$

23. From $x^3-3x^2+6x-10$ take x^3-4x^2+8x-9; x^2-2x-1

24. From $a+\frac{1}{2}b+1$ take $\frac{1}{2}a+b+\frac{1}{2}$ Diff. $=\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}$

25. From $\frac{2}{3}x^2-\frac{5}{4}xy+\frac{3}{2}y^2$ take $-\frac{1}{3}x^2-\frac{1}{4}xy-\frac{1}{2}y^2$; $x^2-xy+2y^2$

[*Exercises* D, 1...18, p. 20.]

20. Since $a - b$ *added to* $a + b$ makes $2a$, and $a - b$ *taken from* $a + b$ leaves $2b$, where a and b represent any two quantities whatever, we can already deduce this general statement, viz. *That the difference of any two quantities added to their sum is equal to twice the greater, and the difference taken from the sum is equal to twice the smaller, quantity.* Thus we are at once enabled to solve such questions as the following:—

PROB. 1. The sum of two numbers is 100, and their difference is 50, what are the numbers?

By our Rule, *twice* the greater number $= 100 + 50 = 150$,

$\therefore$ the greater No. $= 75$.

And since the *difference* of the numbers is 50,

$\therefore$ the lesser $= 75 - 50 = 25$.

$\therefore$ the numbers required are 75 and 25: which, upon trial, will answer.

PROB. 2. The united ages of a man and his wife make 77 years, and one is 7 years older than the other; what is the age of each?

Twice the age of the older $= 77 + 7 = 84$, by the rule,

$\therefore$ the age of the older is 42 years;

and $\therefore$ the age of the other $= 42 - 7 = 35$ years.

PROB. 3. Divide the fraction $\frac{1}{2}$ into two parts, so that one shall exceed the other by $\frac{1}{4}$.

Here the *sum* of the two parts $= \frac{1}{2}$,

and the *difference* .... $= \frac{1}{4}$,

$\therefore$ *twice* the greater part $= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, by the rule,

$\therefore$ the greater part $= \frac{1}{2}$ of $\frac{3}{4} = \frac{3}{8}$.

Also, *twice* the smaller part $= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, by the rule,

$\therefore$ the smaller part $= \frac{1}{2}$ of $\frac{1}{4} = \frac{1}{8}$.

Hence the two parts required are $\frac{3}{8}$, and $\frac{1}{8}$.

[*Exercises* D, 19, 20.]

EXERCISES. D.

(1) From a take $b-x$.

(2) From $a+b-c-d$ take $a-b+c-d$.

(3) From $6a-b-c$ take $a-b+2c$.

(4) From $8a+x-5b-5c$ take $x+2b-5c$.

(5) From $3x+2y-5z$ take $2x+3y+4z$.

(6) From $2ax+by-c$ take $ax-by+c$.

(7) From $3bc-ab+a$ take $2bc+ab-a$.

(8) From $xy+x^2+y^2$ take $xy-x^2+y^2$.

(9) From $2xy+3x^2+4y^2$ take $xy-2x^2-y^2$.

(10) From $2mn+5m-3n$ take $mn+m+n$.

(11) From $-2xy+mx-py$ take $-3xy-2mx-py$.

(12) From $5abc-2ab-3ac$ take $2abc+ab-ac+1$.

(13) From $a^2-b^2+c^2$ take $a^2-2b^2-2c^2$.

(14) From $4ax-3a^2+2x^2$ take $2ax-a^2+4x^2$.

(15) From $3a^2b+2a^2c-5c^2$ take $a^2b-a^2c-7c^2$.

(16) From $2xy+3a-a^2b+5$ take $2a-a^2b+6$.

(17) From $\frac{2}{3}ax-\frac{1}{3}xy+\frac{2}{3}$ take $\frac{1}{3}ax+\frac{2}{3}xy-\frac{1}{3}$.

(18) From $a+b-c$ take $\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}c$.

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(19) The united ages of a father and his son make 60 years, and the father was 30 years old when the son was born, what is the age of each?

(20) Divide 1 into two fractional parts, so that one part shall exceed the other by $\frac{1}{2}$.

MULTIPLICATION.

21. *To multiply one single term by another.*

RULE. *Write the* LETTERS (*of the* SINGLE TERMS *to be multiplied together*) *side by side, as* FACTORS *of the required* PRODUCT: *and multiply the* NUMERICAL COEFFICIENTS *together for the new coefficient**, *prefixing the sign + if the terms to be multiplied together have both one sign, and −, if they have different signs.*

* The learner must bear in mind that in every case where a quantity appears *without* a numerical coefficient, the coefficient 1 is understood.—See Art. 7.

For, 1st. If the quantities be both *positive*, as $2a$, and $3b$, their *product*, by Art. 4, is *equal to* $2a \times 3b$. Now $2a \times 3b = 2 \times a \times 3 \times b$, and $a \times 3 = 3 \times a$, (Art. 5), $\therefore$ the product required $= 2 \times 3 \times a \times b = 6ab$, since $2 \times 3 = 6$.

2nd. If one of the quantities be *negative*, or the product is required of $2a$ times $-3b$, or $-2a$ times $3b$, in either case the meaning can only be, that $3b$ is to be *subtracted* $2a$ times, and it is clear that this will differ from $3b$ to be *added* $2a$ times only in the *sign* of the product, $\therefore$ the product is $-6ab$.

3rd. If *both* quantities be *negative*, as $-2a$ and $-3b$, $-3b$ is to be *subtracted* $2a$ times, that is, $-6ab$ is to be *subtracted*; but *subtracting* $-6ab$ is the same as *adding* $+6ab$, (Art. 19), $\therefore$ the product is $+6ab$.

Hence it appears, that

$$\begin{array}{llll} +3b & \text{multiplied by} & +2a & \text{produces } +6ab, \\ -3b & \ldots\ldots\ldots\ldots\ldots\ldots & +2a & \ldots\ldots\ldots\ldots -6ab, \\ +3b & \ldots\ldots\ldots\ldots\ldots\ldots & -2a & \ldots\ldots\ldots\ldots -6ab, \\ -3b & \ldots\ldots\ldots\ldots\ldots\ldots & -2a & \ldots\ldots\ldots\ldots +6ab. \end{array}$$

And since the same reasoning will apply to any other quantities besides those here used, we embrace all cases in the Rule above stated.

Exs. $2x \times 5y = 10xy$; $-3 \times 5a = -15a$; $7m \times -n = -7mn$.

$2ab \times 3ac = 6aabc$, or $6a^2bc$; $-7axy \times 4abc = -28a^2bcxy$;

$2a \times 3b \times 4c = 6ab \times 4c = 6 \times 4 \times abc = 24abc$.

[*Exercises* E, 1...6, p. 24.]

22. *To multiply a quantity consisting of two or more terms by a single term.*

Rule. *Multiply each term of the multiplicand separately according to the rule stated in Art.* 21, *and the sum of these separate products is the product required.*

Thus, let it be required to multiply $a + b + c +$ &c., by m; then a taken m times is ma, b taken m times is mb, c taken m times is mc, &c., and the sum of these separate products is

$ma + mb + mc +$ &c., which is the product required.

For it is evident, that the *parts* which make up the whole being separately taken m times, and added together, must produce the same as the whole quantity taken m times. Hence the Rule in this case is as above stated.

Ex. 1. $a+b-c$ multiplied by $2=2a+2b-2c$.
Ex. 2. $a-b+c$ .... $-2=-2a+2b-2c$.
Ex. 3. $a-b+c$ .... $d=ad-bd+cd$.
Ex. 4. $a-b+c$ .... $-d=-ad+bd-cd$.
Ex. 5. $ax+by$ .... $c=acx+bcy$.
Ex. 6. $ax+by-cz$ .... $2p=2apx+2bpy-2cpz$.
Ex. 7. $2a+3b-4c$ .... $2x=4ax+6bx-8cx$.
Ex. 8. $ax+by$ .... $ax=a^2x^2+abxy$.
Ex. 9. $ax+by$ .... $-by=-abxy-b^2y^2$.
Ex. 10. $7x-4y+6$ .... $3x=21x^2-12xy+18x$.
Ex. 11. $6x^2-13x+1$ .... $5=30x^2-65x+5$.
Ex. 12. x^2-px+q .... $px=px^3-p^2x^2+pqx$.
Ex. 13. $\frac{1}{2}ab+\frac{3}{2}cd$ .... $4ab=2a^2b^2+6abcd$.
Ex. 14. $mx+3-y$ .... $-\frac{1}{2}n=-\frac{1}{2}mnx-\frac{3}{2}n+\frac{1}{2}ny$.

[*Exercises* E, 7...16, p. 25.]

23. *To multiply one quantity by another, when both consist of two or more terms.*

RULE. *Multiply each term of the multiplicand by each term of the multiplier, according to the rule for single terms, and the* SUM *of these separate products will be the product required.*

For, let it be required to multiply $a+b$ by $c+d$; this means that $a+b$ is to be taken $c+d$ times, that is, c times *and* d times. Now $a+b$ taken c times produces, by rule of Art. 22, $ac+bc$; and $a+b$ taken d times produces, by the same rule, $ad+bd$; $\therefore$ $a+b$ taken c times *and* d times, that is, $c+d$ times, produces $ac+bc+ad+bd$, which is the product required.

Or, if the quantities be $a+b$, and $c-d$, $a+b$ multiplied by $c-d$ means that $a+b$ is to be taken d times less than c times. Now $a+b$ taken c times produces $ac+bc$; but this is too much by d times $a+b$, that is, by $ad+bd$; $\therefore$ $ad+bd$ is to be *subtracted* from $ac+bc$. Hence the product required is $ac+bc-ad-bd$, following the rule of subtraction in Art. 19.

Or, if the quantities be $a-b$, and $c-d$, the product of these is, as in the last case, c times $a-b$ wanting d times $a-b$, that is, $ad-bd$ *subtracted* from $ac-bc$, which leaves $ac-bc-ad+bd$, (changing the *signs* in the quantity to be subtracted, according to rule).

Hence it appears, that

$a+b$ multiplied by $c+d$ produces $ac+bc+ad+bd$,
$a+b$ .... $c-d$ .... $ac+bc-ad-bd$,
$a-b$ .... $c-d$ .... $ac-bc-ad+bd$;

therefore the Rule in this case is as above stated.

When either multiplicand, or multiplier, or both, consist of more than *two* terms, the rule is not altered, as may easily be seen.

EXAMPLES.

1. Mult. $x+1$ by $x+2$

$$\begin{array}{ll} \text{Prod}^{\text{t}}\text{. by } x = & x^2+x \\ \ldots \text{ by } +2 = & \quad +2x+2 \\ \hline \text{Whole prod}^{\text{t}}\text{.} = & x^2+3x+2 \end{array}$$

2. Mult. 21, or $20+1$ by 19, or $20-1$

$$\begin{array}{ll} 189 & 400+20 \\ 21 & \quad -20-1 \\ \hline \text{Prod}^{\text{t}}\text{.} = 399, & \text{or } 400-1 \end{array}$$

3. Mult. $2+a$ by $3-b$

$$\begin{array}{ll} \text{Prod}^{\text{t}}\text{. by } 3 = & 6+3a \\ \ldots \text{ by } -b = & \quad -2b-ab \\ \hline \text{Whole prod}^{\text{t}}\text{.} = & 6+3a-2b-ab \end{array}$$

4. Mult. $a+b$ by $a+b$

$$\begin{array}{ll} \text{Prod}^{\text{t}}\text{. by } a = & a^2+ab \\ \ldots \text{ by } +b = & \quad +ab+b^2 \\ \hline \text{Whole prod}^{\text{t}}\text{.} = & a^2+2ab+b^2 \end{array}$$

5. Mult. $a-b$ by $a-b$

$$\begin{array}{ll} \text{Prod}^{\text{t}}\text{. by } a = & a^2-ab \\ \ldots \text{ by } -b = & \quad -ab+b^2 \\ \hline \text{Whole prod}^{\text{t}}\text{.} = & a^2-2ab+b^2 \end{array}$$

6. Mult. $x-2y$ by $2x+3y$

$$\begin{array}{ll} \text{Prod}^{\text{t}}\text{. by } 2x = & 2x^2-4xy \\ \ldots \text{ by } +3y = & \quad +3xy-6y^2 \\ \hline \text{Whole prod}^{\text{t}}\text{.} = & 2x^2-xy-6y^2 \end{array}$$

7. Mult. $2a+3b-4c$ by $a+b-c$

$$\begin{array}{ll} \text{Prod}^{\text{t}}\text{. by } a = & 2a^2+3ab-4ac \\ \ldots \text{ by } +b = & \quad +2ab+3b^2-4bc \\ \ldots \text{ by } -c = & \quad -2ac-3bc+4c^2 \\ \hline \text{Whole prod}^{\text{t}}\text{.} = & 2a^2+5ab-6ac+3b^2-7bc+4c^2 \end{array}$$

[*Exercises* E, 17...31, p. 25.]

24. *To multiply* POWERS *of the same quantity together.*

RULE. *Powers of the same quantity are multiplied together by adding the indices of the powers together.*

Thus $a^2 \times a^3 = a^5$; for $a^2 = aa$ (Art. 9), and $a^3 = aaa$, $\therefore\ a^2 \times a^3 = aa \times aaa = aaaaa$, or a^5.

In the same manner it may be shewn that $a^6 \times a^{10} = a^{16}$; and so on for other powers, always taking the *sum* of the *indices*. To prove this *generally*, viz. that

$a^m \times a^n = a^{m+n}$, whatever numbers m and n may stand for, we have, by Definition (Art. 9),

$$a^m = a \,.\, a \,.\, a \,.\, \&c. \text{ to } m \textit{ factors},$$

$$\text{and } a^n = a \,.\, a \,.\, a \,.\, \&c. \text{ to } n \textit{ factors},$$

$$\therefore\ a^m \times a^n = a \,.\, a \,.\, a \,.\, \&c. \text{ to } m \textit{ factors} \times a \,.\, a \,.\, \&c. \text{ to } n \textit{ factors},$$

$$= a \,.\, a \,.\, a \,.\, \&c. \text{ to } m + n \textit{ factors},$$

$$= a^{m+n}, \text{ by Definition.}$$

The reasoning and the rule are the same, if for a we write $a + b$, or $a + b + c$, or any other quantity; that is, the *powers* of such quantities are multiplied together by adding the *indices* of the powers together. Thus the 2nd power of $a + b$ multiplied by the 3rd power of the same quantity will produce the 5th power of that quantity.

Ex. 1. $2x^2 \times 3x^3 = 2 \times 3 \times x^2x^3 = 6x^5$.

Ex. 2. $7ax \times 2axy = 7 \times 2 \times aaxxy = 14a^2x^2y$.

Ex. 3. $5a^6b \times abc = 5a^6abbc = 5a^7b^2c$.

Ex. 4. $3x^2y^2z^2 \times 4x^3y^2z = 3 \times 4 \times x^2x^3y^2y^2z^2z = 12x^5y^4z^3$.

Ex. 5. $mnx^3y \times -py = -mnpx^3yy = -mnpx^3y^2$.

Ex. 6. $-4ab^2cx \times -2acx^2y = 8aab^2ccxx^2y = 8a^2b^2c^2x^3y$.

Ex. 7. $2a^m \times 3a^3 = 2 \times 3 \times a^m a^3 = 6a^{m+3}$.

Ex. 8. $ax^m \times bx^n = abx^m x^n = abx^{m+n}$.

Ex. 9. $ax^m \times bx^n \times cx^p = abcx^m x^n x^p = abcx^{m+n+p}$.

Ex. 10. $2ax \times -3by \times -a^3x^2y = 2 \times -3 \times -1 \times a^4bx^3y^2 = 6a^4bx^3y^2$.

[*Exercises* E, 32...40.]

EXERCISES. E.

Multiply

(1) axy by b.

(2) $3mn$ by $-p$.

(3) $-2xy$ by $4a$.

(4) $-2xy$ by $-4a$.

(5) $\frac{1}{2}ab$ by $2c$.

(6) $3mn$ by mp.

Multiply

(7) $m+n-p$ by 3.

(8) $ax+bx^2$ by p.

(9) $ad+2bd$ by $2a$.

(10) $4a^2-2axy$ by ax.

(11) $3x-2xy+6$ by $-xy$.

(12) $1-2ax+3bx^2$ by $-3n$.

(13) $2ab-3ac+5bd$ by $-2x$.

(14) $2xy-3$ by $7x$.

(15) $2ax+by-cz$ by $2xyz$.

(16) $2a^2-bx+d$ by by.

---

(17) $a+x$ by $b+y$.

(18) $6x+4$ by $x-1$.

(19) $x-4$ by $x+3$.

(20) $2x-5$ by $3x-2$.

(21) $1-x$ by $x+1$.

(22) $1-x$ by $x-2x^2$.

(23) $ax+by$ by $2x-y$.

(24) $a+2x$ by $a-3x$.

(25) $7x-1$ by $5x-4$.

(26) $2ax-3by$ by $4y-3x$.

(27) $1-2mn$ by $2m+n$.

(28) a^2-bc by $ac-b^2$.

(29) $1+2x+3y$ by $x-y$.

(30) $a+x-y$ by $b-y$.

(31) $ac-bc+ad$ by $2a-b$.

---

(32) a^3+a^2+a+1 by $a-1$.

(33) $x^3+ax^2+a^2x+a^3$ by $x-a$.

(34) $4x^2-6x+9$ by $2x+3$.

(35) $4+2x+x^2$ by $4-2x+x^2$.

(36) a^3-2x^2 by a^3-x^2.

(37) $x^3+3x^2+9x+27$ by $x-3$.

(38) $2a^4x^2+3b^2y$ by $2a^4x^2-3b^2y$.

(39) $2a^2-3ab+b^2$ by $2a^2+3ab-b^2$.

(40) a^6+a^5-a-1 by $1-a+a^2-a^3+a^4$.

DIVISION.

The words *Dividend*, *Divisor*, and *Quotient*, have the same meaning here as in Common Arithmetic.

To *divide* one quantity by another is to find *how often* the latter is contained in the former, that is, the *Quotient:* and it follows from the nature of Division that the *Quotient* is always that quantity which being multiplied by the *Divisor* will produce the *Dividend.*

25. *To divide one single term by another.*

RULE. *Split the dividend into two factors, making the divisor one of them, and the other factor is the quotient.*

For, since *Quotient* × *Divisor* = *Dividend*, it is clear that, if we can form the *Dividend* into two *factors*, one of which is the same as the *Divisor*, the other *factor* is the *Quotient*. Thus, if it be required to divide $3x$ by x, since the *co-factor* of x in $3x$ is 3, 3 is the *quotient*. Or if it be required to divide $3x$ by 3, x is the *co-factor* of 3 in $3x$; therefore x is the *quotient* in this case.

Hence, when one single term contains another exactly, to divide one by the other the Rule is as above stated.

Ex. 1. To divide $6abc$ by $2ab$.

Here $6abc = 2ab \times 3c$, therefore, by the rule, $3c$ is the *Quotient*.

Ex. 2. To divide $10xy$ by $2y$.

Here $10xy = 2y \times 5x$, therefore $5x$ is the *Quotient*.

Ex. 3. To divide $-7axy$ by $7ax$.

Here $-7axy = 7ax \times -y$, therefore $-y$ is the *Quotient*.

Ex. 4. To divide $6mnpr$ by $-mpr$.

Here $6mnpr = -mpr \times -6n$, therefore $-6n$ is the *Quotient*.

Ex. 5. To divide $-14a^2bc$ by $-2ab$.

Here $-14a^2bc = -2ab \times 7ac$, therefore $7ac$ is the *Quotient*.

Ex. 6. To divide $-8a^2b^3c^4$ by $4abc$.

Here $-8a^2b^5c^4 = 4abc \times -2ab^2c^3$, therefore $-2ab^2c^3$ is the *Quotient*.

Ex. 7. To divide $5a^5b^3c^5$ by a^2bc^2.

Here $5a^5b^3c^5 = a^2bc^2 \times 5a^3b^2c^3$, therefore $5a^3b^2c^3$ is the *Quotient*.

Ex. 8. To divide $21mn^3p$ by $\frac{3}{2}mnp$.

Here $21mn^3p = \frac{3}{2}mnp \times 14n^2$, therefore $14n^2$ is the *Quotient*.

Observe, any example may be stated differently, and perhaps with more clearness, in the following manner. Take Ex. 1 above.

How many *times* does $6abc$ contain $2ab$?

Here $6abc = 3c \times 2ab$, that is, $3c$ *times* $2ab$, (Art. 4), therefore $3c$ is the number of *times* required, or the Quotient.

[*Exercises* F, 1...12, p. 30.]

26. *To divide a quantity consisting of two or more terms by a single term.*

RULE. *Divide each term of the dividend separately by the divisor according to the rule in the preceding Article, and the sum of the several quotients is the quotient required.*

For, since $a + b + c +$ &c. *multiplied* by m produces

$$ma + mb + mc + \&c. \text{ (Art. 22),}$$

$\therefore$ $ma + mb + mc +$ &c., *divided* by m, gives $a + b + c +$ &c., that is, $ma \div m + mb \div m + mc \div m +$ &c. Hence the Rule in this case is as above stated.

Ex. 1. To divide $ab + 2ac - 3ad$ by a.

Here $ab \div a = b$, $+ 2ac \div a = + 2c$, $- 3ad \div a = - 3d$, therefore the whole quantity divided by a is $b + 2c - 3d$, or $b + 2c - 3d$ is the Quotient required.

Ex. 2. To divide $mx + nx^2 - pxy$ by x.

Here $mx \div x = m$, $+ nx^2 \div x = + nx$, $- pxy \div x = - py$, therefore the whole quantity divided by x is $m + nx - py$, or $m + nx - py$ is the Quotient required.

Ex. 3. To divide $4a^2x^2 - 6a^2bx + 2ax^3$ by $2ax$.

Here $4a^2x^2 \div 2ax = 2ax$, $- 6a^2bx \div 2ax = - 3ab$, $+ 2ax^3 \div 2ax = + x^2$,

$\therefore$ $2ax - 3ab + x^2$ is the Quotient required.

[*Exercises* 13...18, p. 30.]

27. *To divide one quantity by another when the* DIVISOR *consists of two or more terms.*

RULE. 1st. Arrange the terms of *both* divisor and dividend according to the *powers* of some one letter, (if this be not already done) that is, beginning with the highest power and going *regularly* down to the lowest, or *vice versâ*, (it matters not which, only it must be the same in both divisor and dividend).

2nd. Find how often the first term of the divisor is contained in the first term of the dividend, by the rule for single terms, (Art. 25), and put this quotient for a *part* of the quotient required.

3rd. Multiply the whole divisor by this quotient, and place the product immediately under, and *subtract* it from, the dividend.

4th. Taking the remainder thus found as a *new dividend*, repeat the operation, again and again, until either 0 remains, or some quantity *less than* the divisor. The *sum* of the several quotients thus found will be the quotient required.

In every respect this rule is the same as that for Long Division in *Arithmetic,* and grounded upon the same reasons. Thus to divide three hundred and eighty four by thirty-two, we first arrange divisor and dividend according to powers of 10, beginning with the highest—the divisor being written 32, which means $3\times10+2$, and the dividend 384, which means $3\times10^2+8\times10+4$. Then we see how often the first term of the divisor, 3×10, or 30, is contained in the first term of the dividend, 3×10^2, or 300, which is 10 times; we therefore put 10 as a *part* of the quotient. Then 10 times 32, or 320, subtracted from 384, leaves for first remainder 64. Using 64 for a new dividend, 32 is contained in it 2 times exactly, leaving no remainder. Hence the whole quotient is $10+2$, or 12.

Ex. 1. To divide $ac+bc+ad+bd$ by $a+b$.

Here the divisor and dividend, arranged according to powers of a, are $a+b$, and $ac+ad+bc+bd$. The succeeding operation, according to the rule, is represented as follows:

$$
\begin{array}{l}
a+b)\ ac+ad+bc+bd\ (c+d \\
\qquad\ \ \underline{ac+bc\qquad\quad} \\
\qquad\qquad\ \ +ad+bd \\
\qquad\qquad\ \ \underline{+ad+bd} \\
\qquad\qquad\qquad\ \ 0
\end{array}
$$

$\therefore\ c+d$ is the Quotient required.

In this Example we first seek how often a is contained in ac which is c times, and we put c as a *part* of the *quotient* to the right hand: then multiply the divisor, $a+b$, by c, which produces $ac+bc$; then subtract this product from the dividend, which leaves the remainder $+ad+bd$. We

proceed with this remainder as a new *dividend*, and repeat the same operation, by which we obtain $+d$ for another part of the quotient, with *no* remainder. Hence $c+d$ is the *whole* Quotient.

Ex. 2. Divide a^2+b^2-2ab by $a-b$.

Here the divisor and dividend, arranged according to powers of a, are $a-b$ and $a^2-2ab+b^2$. Then we proceed thus:

$$
\begin{array}{l}
a-b\;)\;a^2-2ab+b^2\;(\;a-b, \text{ the Quotient.} \\
\qquad\quad\; a^2-ab \\
\qquad\quad\; \overline{\quad\;\; -ab+b^2} \\
\qquad\qquad\;\; -ab+b^2 \\
\qquad\qquad\;\; \overline{\qquad 0 \qquad}
\end{array}
$$

1st we seek how often a is contained in a^2, which gives a, the 1st term of the quotient; $a-b$ multiplied by a gives a^2-ab; this subtracted from $a^2-2ab+b^2$ leaves $-ab+b^2$. Then we repeat the same process with $-ab+b^2$ for a dividend; we seek how often a is contained in $-ab$, which gives $-b$, the second term of the quotient: $a-b$ multiplied by $-b$ gives $-ab+b^2$, which subtracted from the new dividend, leaves 0. Hence $a-b$ is the whole Quotient required.

Ex. 3. Divide $2a^2+3b^2+4c^2+5ab-6ac-7bc$ by $a+b-c$.

Here arranging according to powers of a,

$$
\begin{array}{l}
a+b-c\;)\;2a^2+5ab-6ac+3b^2-7bc+4c^2\;(\;2a+3b-4c \\
\qquad\qquad\; 2a^2+2ab-2ac \\
\qquad\qquad\; \overline{\qquad\quad +3ab-4ac+3b^2-7bc+4c^2} \\
\qquad\qquad\qquad\;\; +3ab \qquad\quad +3b^2-3bc \\
\qquad\qquad\qquad\;\; \overline{\qquad\qquad -4ac-4bc+4c^2} \\
\qquad\qquad\qquad\qquad\qquad\;\; -4ac-4bc+4c^2 \\
\qquad\qquad\qquad\qquad\qquad\;\; \overline{\qquad\quad 0 \qquad\quad}
\end{array}
$$

$\therefore\; 2a+3b-4c$ is the Quotient required.

Ex. 4. Divide $64 - a^6$ by $2 - a$.

$$
\begin{array}{l}
2 - a\,)\ 64 - a^6\ (\,32 + 16a + 8a^2 + 4a^3 + 2a^4 + a^5 \\
\qquad\ \ 64 - 32a \\
\hline
\qquad\qquad 32a - a^6 \\
\qquad\qquad 32a - 16a^2 \\
\hline
\qquad\qquad\qquad 16a^2 - a^6 \\
\qquad\qquad\qquad 16a^2 - 8a^3 \\
\hline
\qquad\qquad\qquad\qquad 8a^3 - a^6 \\
\qquad\qquad\qquad\qquad 8a^3 - 4a^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad 4a^4 - a^6 \\
\qquad\qquad\qquad\qquad\qquad 4a^4 - 2a^5 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad 2a^5 - a^6 \\
\qquad\qquad\qquad\qquad\qquad\qquad 2a^5 - a^6 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

$\therefore\ 32 + 16a + 8a^2 + 4a^3 + 2a^4 + a^5$ is the Quotient required.

[*Exercises* F, 19...31.]

EXERCISES. F.

Divide

(1) $7x$ by 7.

(2) $7x$ by x.

(3) $7ax$ by a.

(4) $7ax$ by $7x$.

(5) $3abx$ by ab.

(6) $3abc$ by $3bc$.

(7) $-axy$ by x.

(8) axy by $-x$.

(9) $6a^2mn$ by $-2mna$.

(10) $14a^3xy^2$ by $7a^2y$.

(11) $-7mn^2px$ by $\frac{1}{2}mnp$.

(12) $-\frac{2}{3}abx^2y$ by $-\frac{1}{3}axy$.

(13) $3ac - 2abd$ by a.

(14) $4ac - 2abd$ by $2a$.

(15) $8x^2 - 6xy$ by $-2x$.

(16) $3bc + 24abc^2 - 6b^2c^2$ by $3bc$.

(17) $4a^2x^2 - 8abx - 2ax$ by $-2ax$.

(18) $a^3x^2 - 5abx^3 + 6ax^4$ by ax^2.

(19) $x^2 + 3x + 2$ by $x + 2$.

(20) $ac - bc + ad - bd$ by $a - b$.

(21) $6 + 3a - 2b - ab$ by $2 + a$.

(22) $4a^2 - 15x^2 - 4ax$ by $2a + 3x$.

Divide

(23) $2a^2 + a - 6$ by $2a - 3$.

(24) $2ab + 6abc - 8abcd$ by $1 + 3c - 4cd$.

(25) $3x^2 + 16x - 35$ by $x + 7$.

(26) $3x^4 + 14x^3 + 9x + 2$ by $x^2 + 5x + 1$.

(27) $ab + 2a^2 - 3b^2 - 4bc - ac - c^2$ by $2a + 3b + c$.

(28) $15a^4 + 10a^3x + 4a^2x^2 + 6ax^3 - 3x^4$ by $3a^2 - x^2 + 2ax$.

(29) $qp^3 + 3p^2q^2 - 2pq^3 - 2q^4$ by $p - q$.

(30) $a^2x^3 + a^5 - 2abx^3 + b^2x^3 + a^3b^2 - 2a^4b$ by $ax - bx + a^2 - ab$.

(31) $32x^5 + 243$ by $2x + 3$.

GREATEST COMMON MEASURE.

28. DEF. That which will *divide* a quantity without a remainder is called a "*measure*" of that quantity. Consequently that which will divide each of *two or more* quantities is called the "*Common Measure*" of those quantities, being a measure common to them all; and the *Greatest Common Divisor* is therefore the "*Greatest Common Measure*". In fact, *measure* is only another word for *divisor*, restricting the latter word to such quantities only as will divide *without remainder*.

Thus 5 is a *measure* of 15, because it will *divide* 15 without remainder; it is also a *measure* of 25 for the same reason: therefore 5 is a *common measure* of 15 and 25. Similarly 2 is a *common measure* of 8 and 12; so also is 4: and 4 is greater than 2. Therefore, as there is no other *common measure* of 8 and 12, except 2 and 4, their *Greatest Common Measure* is 4.

Again, since $2a$ is divisible by a *without remainder*, and so also is $3a$, a is a common measure of $2a$ and $3a$; and as there is no other common measure, it is therefore the *Greatest Common Measure* of $2a$ and $3a$.

It is evident, then, that a *measure* of any quantity must be a *factor* of that quantity; so that if we can split up a quantity into *all* the simple factors by which it is made up, we can then *see* before us *all the measures* of that quantity, and by doing the same with another quantity, we can at

once say which measures are *common* to both. Either the greatest *factor*, or greatest *product of two or more factors, common to both,* will be the "*Greatest Common Measure*" of the two quantities.

29. To split up any *number* into its component *factors*, we try all the numbers 2, 3, 4, 5, 6, &c. in order as *divisors*, to see if they are *measures*, and repeat each of them, which we find to *measure*, as long as it remains a *measure* of the quotient so obtained: thus taking the number 189, we write our operation as follows: (2 will not divide it, but 3 will, so we begin with 3,)

$$\begin{array}{r|r}
3 & 189 \\ \hline
3 & 63 \\ \hline
3 & 21 \\ \hline
7 & 7 \\ \hline
 & 1
\end{array} \quad \therefore\ 189 = 3 \times 3 \times 3 \times 7.$$

Now take 224,

$$\begin{array}{r|r}
2 & 224 \\ \hline
2 & 112 \\ \hline
2 & 56 \\ \hline
2 & 28 \\ \hline
2 & 14 \\ \hline
7 & 7 \\ \hline
 & 1
\end{array} \quad \therefore\ 224 = 2 \times 2 \times 2 \times 2 \times 2 \times 7.$$

In the first case, 189 not being divisible by 2, we began with 3, and *repeated* it until we could no longer divide *without remainder;* then we passed over 4, 5, 6, because none of them would divide *without remainder*.

In the second case, 224 was divisible by 2 five times successively, but then only by 7.

Hence, the *measures* of 189 are 3, 3, 3, and 7.
.... 224 .... 2, 2, 2, 2, 2, and 7; of which 7 only is *common* to both; therefore 7 is a *common measure*, and also the *greatest common measure* of 189 and 224.

Ex. To find the G.C.M.* of 385 and 396.

$$\begin{array}{r|l} 5 & 385 \\ \hline 7 & 77 \\ \hline 11 & 11 \\ \hline & 1 \end{array} \qquad \begin{array}{r|l} 2 & 396 \\ \hline 2 & 198 \\ \hline 3 & 99 \\ \hline 3 & 33 \\ \hline 11 & 11 \\ \hline & 1 \end{array}$$

$\therefore\ 396 = 2\times2\times3\times3\times11.$

$\therefore\ 385 = 5\times7\times11.$

And since 11 is the only *factor common* to both, therefore 11 is the G.C.M. of 385 and 396.

For the *usual* method of finding the G.C.M. of two or more *numbers*, see any treatise on *Arithmetic*, or Wood's Algebra, Art. 19.

30. To split up an *Algebraical* quantity into its component simple factors, can only be learnt by practice: but for quantities of a *single term* the method is obvious enough. Thus $2a^2bc^2 = 2\times aabcc$, $4a^3b^2c = 2\times2\times aaabbc$; and so on: in which form *we see all* the factors of the proposed quantities. Then, supposing the G.C.M. of $2a^2bc^2$, and $4a^3b^2c$, to be required, we *see* that it is the product of the *common* factors, $2,a,a,b,c$, or $2a^2bc$.

Again, to find the G.C.M. of $3a^4x^5y$, and $6a^2bx$; here

$$3a^4x^5y = 3\times aaaaxxxxxy,$$
$$6a^2bx = 2\times3\times aabx,$$

in which the factors common to both are $3, a, a, x,$ and no other factor;

$$\therefore\ \text{G.C.M.} = 3\times aax = 3a^2x.$$

After much practice the Student will abridge the operation in most cases, and it will become more and more a matter of *eye-sight*.

To find the G.C.M. of quantities consisting of *two or more terms* is not needed for the present work, and had better be deferred until the Student is able to take up the larger work of Dr. Wood† with effect.

* By G.C.M. is meant 'greatest common measure' henceforward.

† Wood's *Algebra*, new Edition, by Lund.

EXERCISES. G.

Find the G.C.M. of

(1) 128, and 84.

(2) 125, and 900.

(3) 80, 100, and 140.

(4) ax, and bx.

(5) bx^2, and b^2x^3.

(6) apx^2, and a^2px.

(7) $5a^2bx$, and $20abxy$.

(8) $15a^6b^2$, and $3a^2b^6$.

(9) $9a^4b^2c^6$, and $27a^6b^3c^9$.

(10) $14m^2np^3$, and $7mnp$.

(11) $abxy$, and $2acxy$.

(12) $\frac{4}{5}a^2$, and $\frac{2}{5}ab$.

(13) abd, acd, and bcd.

(14) pxy, x^2y^2, and apx.

LEAST COMMON MULTIPLE.

31. DEF. A *multiple* of a quantity is that which contains the quantity some number of times exactly, that is, which is divisible by it without remainder. Consequently that which each of *two or more* quantities will divide without remainder is a *common multiple* of those quantities; and the least quantity which will do this is the "*Least Common Multiple*".

Thus 15 is a multiple of 5, because it contains 5 three times exactly: 15 is also a multiple of 3, because it contains 3 five times exactly; therefore 15 is a *Common Multiple* of 5 and 3. Similarly 30 is a *Common Multiple* of the same numbers 5 and 3; so also is 45. But 15 is the *least* of such numbers, therefore it is the "*Least Common Multiple*" (L.C.M.) of 5 and 3.

Again, $2ab$ is a *multiple* of a, because it contains a exactly $2b$ times; it is also a *multiple* of b, because it contains b exactly $2a$ times; therefore $2ab$ is a *common multiple* of a and b, but it is not the *Least Common Multiple*, since ab, which is also a *common multiple* of a and b, is *less* than $2ab$.

It is plain, then, that a *multiple* of any quantity must have that quantity for one of its *factors;* and a *common multiple* of *two or more* quantities must have *each* of the quantities as a *factor*, so that the *product* of any number of quantities is always a *common multiple* of them all, but not always the *Least Common Multiple*. Thus of 2, 4, 6, the product of $2\times4\times6$, or 48, is a *common multiple*, but the *Least Common Multiple* is 12.

32. Hence to find the L.C.M. of two or more quantities, split each quantity up into its simple *factors*, and construct a quantity which shall contain every *different* factor found in *all* the proposed quantities, but no factor repeated which is not similarly repeated in some *one* of them. It is obvious then that this new quantity so constructed is a multiple of *each* of the proposed quantities, and also the *least* quantity which contains all of them, that is, the Least Common Multiple of them all.

Ex. 1. Thus, if the L.C.M. of 3, 10, and 6 be required.

1st. $3=3\times1,\qquad 10=2\times5,\qquad 6=2\times3,$

therefore the *different* factors are 3, 1, 2, 5, and no factor is repeated, that is, occurs more than *once*, in any *one* of the proposed numbers,

$\therefore$ the L.C.M. required $=3\times1\times2\times5=30.$

Ex. 2. To find the L.C.M. of 8, 16, 10, and 20.

Here $8=2\times2\times2$, $16=2\times2\times2\times2$, $10=2\times5$, $20=2\times2\times5$, the *different* factors are 2, and 5, and 2 is repeated 4 times in *one* of the proposed numbers;

$\therefore$ the L.C.M. $=2\times2\times2\times2\times5=80.$

Ex. 3. To find the L.C.M. of $2a$, $6ab$, and $8ab$.

Here $2a=2\times a$, $6ab=2\times3\times ab$, $8ab=2\times2\times2\times ab$,

the *different* factors are 2, 3, a, and b, and 2 is repeated 3 times in one of the quantities,

$\therefore$ the L.C.M. $=2\times2\times2\times3\times ab=24ab.$

Ex. 4. To find the L.C.M. of $8a^2$, $12a^3$, and $20a^4$.

Here $8a^2=2\times2\times2aa$, $12a^3=2\times2\times3aaa$, $20a^4=2\times2\times5aaaa$, the *different* factors are 2, 3, 5, and a; 2 is repeated 3 times, and a four times;

$\therefore$ L.C.M. $=2\times2\times2\times3\times5aaaa=120a^4.$

The method of finding the L.C.M. of quantities consisting of *two or more terms* is usually given in treatises on Algebra, but it is not suited to the scheme of this work, and is therefore omitted. What we have here introduced is simply with a view to enable the student rightly to understand the next chapter on *Fractions*.

EXERCISES. H.

Find the L.C.M. of

(1)	21, and 24.	(7)	ax, and bx.
(2)	12, 16, and 20.	(8)	ax, and $2xy$.
(3)	4, 7, 8, and 14.	(9)	$2x$, $6x$, and $8x$.
(4)	4, 7, 14, 21, and 24.	(10)	ab, ac, and bc.
(5)	1, 2, 3, 4, 5, 6, 7, 8, 9.	(11)	x^2, y^2, and $2xy$.
(6)	21, 22, 23, and 24.	(12)	bd, c^2d, cd^2, and bc.

FRACTIONS.

Algebraic Fractions are precisely the same in character and signification as Fractions in *Arithmetic*. Thus $\frac{a}{b}$ signifies that the unit or whole is divided into b equal parts, and a of them are taken, a being the *Numerator* and b the *Denominator*, where a and b are any quantities, that is, general numbers.

33. *To shew that* $\frac{\mathrm{a}}{\mathrm{b}}$ *is equal to the* b^{th} *part of* a.

The meaning of $\frac{a}{b}$, according to the definition of a 'fraction', is that the unit is divided into b equal parts, and a of them are taken to make the quantity represented by $\frac{a}{b}$. Now when the unit is thus divided, it is clear that each part is the b^{th} part of the unit; and $\frac{a}{b}$ is a such parts, that is, a times the b^{th} part of 1; but the b^{th} part of 1, repeated a times, is clearly the same as the b^{th} part of $1+1+1+$ &c. to a terms (Art. 26), and $1+1+$&c. to a terms is a, therefore $\frac{a}{b}$ is equal to the b^{th} part of a.

34. *If the numerator and denominator of a fraction be both multiplied by the same quantity, the* VALUE *of the fraction is not altered.*

Thus $\frac{a}{b}=\frac{2a}{2b}=\frac{3a}{3b}=\text{\&c.}=\frac{na}{nb}$. For $\frac{2a}{2b}$ signifies that the

unit is divided into $2b$ equal parts, and $2a$ of them are taken. Now when the unit is divided into $2b$ equal parts, it is clear that each part is only *half* as great as when the unit was divided into b equal parts; and therefore a of the latter parts are together equal to $2a$ of the former, that is

$$\frac{a}{b}=\frac{2a}{2b}.$$

By similar reasoning it will appear, that $\frac{a}{b}=\frac{3a}{3b}=\frac{na}{nb}$, where n stands for any number whatever, each part in $\frac{na}{nb}$ being $\frac{1}{n}$th of each part in $\frac{a}{b}$, but n times as many being taken of the former parts as of the latter, which preserves the equality.

35. Hence also, since $\frac{na}{nb}=\frac{a}{b}$, *if the numerator and denominator of a fraction be both divided by the same quantity, the* VALUE *of the fraction is not altered.*

Ex. 1. $\frac{a}{b}=\frac{a\times c}{b\times c}=\frac{ac}{bc}$.

Ex. 2. $\frac{a}{b}=\frac{a\times df}{b\times df}=\frac{adf}{bdf}$.

Ex. 3. $\frac{a-x}{x}=\frac{2a-2x}{2x}$.

Ex. 4. $\frac{a-x}{x}=\frac{a^2-ax}{ax}$.

Ex. 5. $\frac{1-x}{1+x}=\frac{y-xy}{y+xy}$.

Ex. 6. $\frac{3a-b}{2a-3b}=\frac{3ab-b^2}{2ab-3b^2}$.

Ex. 7. $36a=\frac{36a}{1}=\frac{252a}{7}$.

Ex. 8. $\frac{ax-x^2}{2ax}=\frac{a-x}{2a}$.

Ex. 9. $\frac{2ax-2x^2}{2ax}=\frac{a-x}{a}$.

Ex. 10. $\frac{a^2+ab}{a^2-ab}=\frac{a+b}{a-b}$.

Ex. 11. $\frac{2a^2b-3ab^2}{7abc}=\frac{2a-3b}{7c}$.

Ex. 12. $\frac{ax-2ax^2}{3ax}=\frac{1-2x}{3}$.

By the last rule fractions are "reduced to lower terms", when they admit of it; for by dividing numerator and denominator by some quantity which will divide them both without remainder the fractions are simplified, as may be seen in the last five Examples, without altering their *value*.

EXERCISES. I.

Reduce the following fractions to lowest terms:—

(1) $\dfrac{2ax}{3x}$.

(2) $\dfrac{4abc}{2ac}$.

(3) $\dfrac{20abx}{15a^2}$.

(4) $\dfrac{3abx^2}{6ax}$.

(5) $\dfrac{75ax^2y^3}{15a^2y^3}$.

(6) $\dfrac{ab^3x}{2ab^4x^2}$.

(7) $\dfrac{mx-nx}{mnx}$.

(8) $\dfrac{2x^2-3x}{5x}$.

(9) $\dfrac{14a^3+21a^2}{7a^2b}$.

(10) $\dfrac{4bc+2c}{2ac}$.

(11) $\dfrac{3ax-2x^2}{2ax-3x^2}$.

(12) $\dfrac{mnp-m^2p+mp^2}{m^2p-mnp+mp^2}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

36. *To add two or more fractions together.*

RULE 1st. *If the fractions have the same denominator, add the numerators together for a new numerator, and retain the common denominator.*

Thus $\dfrac{a}{b}+\dfrac{c}{b}=\dfrac{a+c}{b}$, just as $\dfrac{1}{4}+\dfrac{2}{4}=\dfrac{3}{4}$. For in each of the algebraic fractions the unit is divided into b equal parts, b being the *denominator* of each; and it is clear that a of these parts, or $\dfrac{a}{b}$, added to c of the same parts, or $\dfrac{c}{b}$, gives $a+c$ such parts, or $\dfrac{a+c}{b}$.

Similarly, $\dfrac{a}{b}+\dfrac{c}{b}+\dfrac{d}{b}=\dfrac{a+c+d}{b}$; and so on, whatever be the number of fractions.

RULE 2nd. *If the fractions have not the same denominator, they must be replaced by others which have, without altering their value, by Art. 34, or 35.*

Thus, to add together $\frac{a}{b}$ and $\frac{c}{d}$, which will represent *any* two fractions with *different denominators:* Since, by Art. 34, $\frac{a}{b}=\frac{ad}{bd}$, and $\frac{c}{d}=\frac{bc}{bd}$, $\therefore \frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$, by the 1st case.

Or, if there be three fractions, $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, then since $\frac{a}{b}$ $=\frac{adf}{bdf}$, $\frac{c}{d}=\frac{c\times bf}{d\times bf}=\frac{bcf}{bdf}$, (for $c\times b=bc$, and $d\times b=bd$, by Art. 5) and $\frac{e}{f}=\frac{bd\times e}{bd\times f}=\frac{bde}{bdf}$, $\therefore \frac{a}{b}+\frac{c}{d}+\frac{e}{f}=\frac{adf}{bdf}+\frac{bcf}{bdf}+\frac{bde}{bdf}$ $=\frac{adf+bcf+bde}{bdf}$; and so on, whatever be the number of the fractions.

Hence the rule in this case is, as in Arithmetic, *Multiply the numerator of each fraction by the product of all the denominators except its own; make the sum of these products the new numerator; and multiply all the denominators together for a new denominator.*

Ex. 1. Add together $\frac{a}{x}$, $\frac{b}{x}$ and $\frac{c}{x}$.

Here the denominators being the same, the sum required

$$=\frac{a+b+c}{x}.$$

Ex. 2. Add together $\frac{a}{x}$, and $\frac{b}{2x}$.

Here the denominators are different, but $\frac{a}{x}=\frac{2a}{2x}$,

$$\therefore \text{ the sum } =\frac{2a}{2x}+\frac{b}{2x}=\frac{2a+b}{2x}.$$

Ex. 3. Add together $\frac{1}{2}$, and $\frac{x}{2a}$.

Here $\frac{1}{2}=\frac{a}{2a}$, $\therefore$ the sum $=\frac{a}{2a}+\frac{x}{2a}=\frac{a+x}{2a}$.

Ex. 4. Add together $\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$.

Here $\frac{x}{2}=\frac{3\times 4\times x}{3\times 4\times 2}=\frac{12x}{24}$, $\frac{x}{3}=\frac{2\times 4\times x}{2\times 4\times 3}=\frac{8x}{24}$,

$$\frac{x}{4}=\frac{2\times3\times x}{2\times3\times4}=\frac{6x}{24},\ \therefore \text{ the sum}=\frac{12x}{24}+\frac{8x}{24}+\frac{6x}{24}=\frac{26x}{24}.$$

Ex. 5. Add together $\frac{1}{x}$, $\frac{1}{2x}$, and $\frac{1}{3x}$.

Here $\frac{1}{x}=\frac{1\times2x\times3x}{x\times2x\times3x}=\frac{6x^2}{6x^3}$, $\frac{1}{2x}=\frac{1\times x\times3x}{2x\times x\times3x}=\frac{3x^2}{6x^3}$,

$$\frac{1}{3x}=\frac{1\times x\times2x}{x\times2x\times3x}=\frac{2x^2}{6x^3},$$

$$\therefore \text{ the sum}=\frac{6x^2}{6x^3}+\frac{3x^2}{6x^3}+\frac{2x^2}{6x^3}=\frac{11x^2}{6x^3},$$

or $\frac{11}{6x}$ in lower terms, Art. 35.

This Ex. is treated according to rule; but it is not the method to be adopted in practice. It is sufficiently obvious *at sight,* that we can easily make the denominators of the proposed fractions all alike, without altering the value of each fraction, by adopting $6x$ for the new denominator; for

$$\frac{1}{x}=\frac{6\times1}{6\times x}=\frac{6}{6x},\ \frac{1}{2x}=\frac{3\times1}{3\times2x}=\frac{3}{6x},\ \frac{1}{3x}=\frac{2\times1}{2\times3x}=\frac{2}{6x},$$

$$\therefore \text{ the sum}=\frac{6+3+2}{6x}=\frac{11}{6x}, \text{ as before.}$$

N.B. Since the "*Least Common Multiple*" of the denominators contains each of them a certain number of times, multiplying the numerator and denominator of each fraction by that number, we shall have the fractions with the L.C.M. for a common denominator, and in their *lowest terms.*

Ex. 1. Add together $\frac{x}{2}$, $\frac{x}{3}$, and $\frac{x}{4}$.

Here the L.C.M. of the denominators is 12, in which 2 is contained 6 times, 3 four times, and 4 three times, therefore multiplying numerator and denominator of each fraction by 6, 4, and 3 respectively,

$$\frac{x}{2}=\frac{6x}{12},\quad \frac{x}{3}=\frac{4x}{12},\quad \frac{x}{4}=\frac{3x}{12},$$

$$\therefore \text{ the sum}=\frac{6x}{12}+\frac{4x}{12}+\frac{3x}{12}=\frac{6x+4x+3x}{12}=\frac{13x}{12}.$$

Ex. 2. Add together $\frac{7x}{6}$, $\frac{3x}{5}$, and $\frac{x}{30}$.

Here the L.C.M. of the denominators is 30,

$$\frac{7x}{6}=\frac{35x}{30},\quad \frac{3x}{5}=\frac{18x}{30},$$

$$\therefore \text{ the sum required }=\frac{35x+18x+x}{30}=\frac{54x}{30}=\frac{9x}{5}.$$

Ex. 3. Add together $\frac{x}{2a}$, $\frac{x}{6ab}$, and $\frac{x}{8ab}$.

Here the L.C.M. of the denominators is $24ab$, (Art. 32, Ex. 3) and $24ab$ is $12b$ times $2a$, 4 times $6ab$, 3 times $8ab$;

$$\therefore \frac{x}{2a}=\frac{12bx}{24ab},\quad \frac{x}{6ab}=\frac{4x}{24ab},\quad \frac{x}{8ab}=\frac{3x}{24ab},$$

$$\therefore \text{ the sum }=\frac{12bx+4x+3x}{24ab}=\frac{12bx+7x}{24ab}.$$

[*Exercises* J, 1...18, p. 42.]

37. *To subtract one fraction from another.*

RULE. *Proceed as in addition, except that one numerator is to be* subtracted *from the other instead of being added to it, to form the new numerator.*

Thus $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$, and $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$.

Ex. 1. Subtract $\frac{2a}{7b}$ from $\frac{9a}{7b}$.

$$\frac{9a}{7b}-\frac{2a}{7b}=\frac{9a-2a}{7b}=\frac{7a}{7b}=\frac{a}{b}.$$

Ex. 2. Subtract $\frac{3x}{24y}$ from $\frac{3x}{4y}$.

$$\frac{3x}{4y}=\frac{6\times 3x}{6\times 4y}=\frac{18x}{24y},$$

$$\therefore \text{ the difference required }=\frac{18x}{24y}-\frac{3x}{24y}=\frac{15x}{24y}=\frac{5x}{8y}.$$

Ex. 3. From $\frac{5ab}{4}$ take $\frac{7ab}{6}$.

Here 12 is the L.C.M. of the denominators,

$$\frac{5ab}{4}=\frac{15ab}{12},\text{ and }\frac{7ab}{6}=\frac{14ab}{12},$$

$\therefore$ difference required $= \frac{15ab}{12} - \frac{14ab}{12} = \frac{ab}{12}$.

N.B. Any quantity, not in a *fractional* form, may be considered and treated as a *fraction* whose denominator is 1; thus $a = \frac{a}{1}$, $x = \frac{x}{1}$, $a - b = \frac{a-b}{1}$, and so on. For $a = \frac{a \times 1}{1} = \frac{a}{1}$, by Art. 34.

[*Exercises* J, 19...32.]

EXERCISES. J.

Add together

(1) $\frac{x}{5}$, $\frac{2x}{5}$, and $\frac{3x}{5}$.

(2) $\frac{2ab}{3}$, and $\frac{ab}{6}$.

(3) $\frac{2a}{3}$, $\frac{a}{3}$, and $\frac{1}{3}$.

(4) $\frac{a+x}{5}$, and $\frac{a-x}{5}$.

(5) $\frac{2x+1}{7}$, and $\frac{4x-5}{7}$.

(6) $\frac{2x+1}{7}$, and $\frac{4x-5}{21}$.

(7) $\frac{1}{a}$, $\frac{2}{a}$, and $\frac{3}{a}$.

(8) $\frac{1}{ab}$, $\frac{2}{ab}$, and $\frac{3}{ab}$.

(9) $\frac{2}{xy}$, $\frac{1}{x^2}$, and $\frac{1}{y^2}$.

(10) $\frac{1}{a}$, $\frac{2}{ab}$, $\frac{3}{abc}$.

(11) x, $\frac{3x-5}{2}$, and $\frac{2x-4}{3}$.

(12) $\frac{x}{6}$, $\frac{7x-6}{3}$, and $\frac{4x+1}{12}$.

(13) $\frac{4x-5}{10}$, $\frac{2x}{5}$, and $\frac{7x+6}{25}$.

(14) $\frac{3}{x}$, $\frac{1}{3x}$, and $\frac{4}{5x}$.

(15) $\frac{4}{5y}$, $\frac{1}{2y}$, and $\frac{3}{4y}$.

(16) $\frac{x}{a}$, $\frac{y}{b}$, and $\frac{z}{c}$.

(17) $\frac{xy-ab}{ab}$, $\frac{xy-bc}{bc}$, and 2.

(18) $\frac{a-b}{ab}$, $\frac{b-c}{bc}$, and $\frac{c-a}{ac}$.

Subtract

(19) $\frac{4x}{5}$ from $\frac{9x}{10}$.

(20) $\frac{7x}{8}$ from x.

(21) $\frac{5x+4}{9}$ from $\frac{10x+17}{18}$.

(22) $\frac{2x-3}{4}$ from $\frac{5x-1}{8}$.

(23) $\frac{3y+x+13}{10}$ from $\frac{3x+y}{5}+1$.

(24) $\frac{15+3x}{x+1}$ from $7+\frac{24}{x+1}$.

Subtract

(25) $\frac{2}{x}+\frac{4}{x}$ from $\frac{3}{x}+\frac{5}{x}$.

(26) $\frac{x}{x+1}$ from $\frac{3x}{x+2}$.

(27) $\frac{2x-7}{21}$ from $\frac{3x+7}{14}$.

(28) $\frac{x}{10}+\frac{4}{25}$ from $\frac{11x-13}{25}$.

(29) $\frac{a}{b+cx}$ from $\frac{a}{b}$.

(30) $\frac{2x}{x+y}$ from $\frac{x+y}{y}$.

(31) $\frac{2}{1+x}$ from $\frac{3+2x}{1+x^2+2x}$.

(32) $\frac{x-y}{x+y}$ from $\frac{x+y}{x-y}$.

MULTIPLICATION AND DIVISION OF FRACTIONS.

38. *To multiply a fraction by a whole number.*

RULE. *Multiply the numerator of the fraction by the whole number for a new numerator, and retain the denominator.*

Thus $c\times\frac{a}{b}=\frac{ac}{b}$; for the unit in $\frac{a}{b}$ and $\frac{ac}{b}$ being divided into the same number of equal parts (since the *denominators* are the same), those parts are all equal to each other, and c times as many parts being taken in the one fraction as in the other, it is clear that the one is c times as great as the other.

Ex. 1. Multiply $\frac{a}{b}$ by 2.

Product $=\frac{2a}{b}$; for twice $\frac{a}{b}$ is $\frac{a}{b}+\frac{a}{b}=\frac{a+a}{b}$ (Art. 36) $=\frac{2a}{b}$.

Ex. 2. Multiply $\frac{ax}{by}$ by m.

$$m\times\frac{ax}{by}=\frac{max}{by}, \text{ the product required.}$$

Ex. 3. Multiply $\frac{a-x}{a+x}$ by 7.

$$\text{Product}=7\times\frac{a-x}{a+x}=\frac{7a-7x}{a+x}.$$

Ex. 4. Multiply $\frac{a-x}{b}$ by $2a$.

$$\text{Product} = 2a \times \frac{a-x}{b} = \frac{2a^2 - 2ax}{b}.$$

[*Exercises* K, 1...15, p. 48.]

39. *To divide a fraction by a whole number.*

Rule. *Divide the numerator of the fraction by the whole number, when that can be done, for a new numerator, and retain the denominator: or multiply the denominator by the number for a new denominator, and retain the numerator.*

Thus $\frac{ac}{b} \div c = \frac{a}{b}$, and $\frac{a}{b} \div c = \frac{a}{bc}$; for since c times $\frac{a}{b} = \frac{ac}{b}$, by Art. 38, the c^{th} part of $\frac{ac}{b}$, that is, $\frac{ac}{b} \div c$, must be $\frac{a}{b}$. And, again, since $\frac{a}{b} = \frac{ac}{bc}$, by Art. 34, and $\frac{ac}{bc} = c$ times $\frac{a}{bc}$, by Art. 38, therefore $\frac{a}{b}$ is c times as great as $\frac{a}{bc}$, and therefore $\frac{a}{bc}$ must be the c^{th} part of $\frac{a}{b}$; or $\frac{a}{b} \div c = \frac{a}{bc}$; which proves the rule.

Ex. 1. Divide $\frac{2a}{b}$ by 2. Quotient $= \frac{a}{b}$, $\because 2a \div 2 = a$.

Ex. 2. Divide $\frac{max}{by}$ by m.

$$\because max \div m = ax, \therefore \text{the Quotient} = \frac{ax}{by}.$$

Ex. 3. Divide $\frac{7a - 7x}{a + x}$ by 7.

$$\because \text{numerator} \div 7 = a - x, \therefore \text{the Quotient} = \frac{a-x}{a+x}.$$

Ex. 4. Divide $\frac{2ab - 2a^2}{c}$ by $2a$.

$$\because 2ab - 2a^2 \text{ divided by } 2a = b - a, \therefore \text{the Quotient} = \frac{b-a}{c}.$$

[*Exercises* K, 21...27, p. 49.]

40. *To multiply one fraction by another fraction.*

Rule. *Multiply the numerators together for a new numerator, and the denominators together for a new denominator.*

Thus $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$. To *prove* the rule, we have $\frac{c}{d}$ to be taken $\frac{a}{b}$ times. Now $\frac{c}{d}$ taken a times is $\frac{ac}{d}$, by Art. 38; but $\frac{a}{b}$ is the b^{th} part of a (Art. 33), therefore $\frac{c}{d}$ is to be taken, not a times, but the b^{th} part of a times. Hence the product required will be the b^{th} part of $\frac{ac}{d}$, that is, $\frac{ac}{d}\div b$, which is $\frac{ac}{bd}$, by Art. 39;

$$\therefore\ \frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d},\ \text{which proves the rule.}$$

Also, since $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$,

$$\therefore\ \frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}=\frac{ac}{bd}\times\frac{e}{f}=\frac{ace}{bdf};$$

and so on, whatever be the number of fractions to be multiplied together.

Ex. 1. Multiply $\frac{2}{a}$ by $\frac{b}{c}$. Product $=\frac{2b}{ac}$.

Ex. 2. Multiply $\frac{a-x}{y}$ by $\frac{6}{x}$.

$$\frac{6}{x}\times\frac{a-x}{y}=\frac{6a-6x}{xy}.$$

Ex. 3. Multiply $\frac{2a}{3y}$ by $\frac{b}{x}$.

$$\frac{2a}{3y}\times\frac{b}{x}=\frac{2a\times b}{3y\times x}=\frac{2ab}{3xy}.$$

Ex. 4. Multiply $\frac{x^2}{a}$ by $\frac{x}{a}$.

$$\text{Product}=\frac{x^2\times x}{a\times a}=\frac{x^3}{a^2}.$$

Ex. 5. Multiply $\frac{ab}{2xy}$ by $\frac{2ab}{5xy}$.

$$\text{Product}=\frac{ab\times 2ab}{2xy\times 5xy}=\frac{2a^2b^2}{10x^2y^2}.$$

The result in the last Example is not in its *lowest* terms, the numerator and denominator being both divisible by 2. This should have been avoided, by observing, before the multiplication was effected, that 2 would be a *common factor* in the numerator and denominator of the *product*, and leaving it entirely out of consideration; for the neglecting of a *factor* common to numerator and denominator is clearly equivalent to dividing numerator and denominator by that factor, which we know does not alter the *value* of a fraction, but merely reduces it to *lower terms*. Similarly, *any number of factors* which we *see* will be common to numerator and denominator of the product may be neglected, if we wish with the least trouble to have the fraction in its lowest terms.

Ex. 6. Multiply $\frac{2x}{3}$ by $\frac{3x}{5}$.

Here $\frac{2x}{3}\times\frac{3x}{5}=\frac{2x^2}{5}$, neglecting the factor 3 common to numerator and denominator of the *product* when found according to rule.

Ex. 7. Multiply $\frac{4x}{5}$ by $\frac{5x}{4}$.

Here the *product*, according to rule, $=\frac{4x\times 5x}{5\times 4}$, and the factors common to numerator and denominator are 4 and 5; omitting these factors the numerator becomes $x\times x$, or x^2, and the denominator 1×1, or 1, therefore the product $=\frac{x^2}{1}$, or x^2. But the student should endeavour to be able to do all this at a single step thus,

$$\frac{4x}{5}\times\frac{5x}{4}=x^2.$$

Ex. 8. Multiply $\frac{2x-5}{4}$ by 4.

Here we say at once that the product is $2x-5$. In fact, 4 times the 4th part of any thing, or quantity, must plainly be the whole thing or quantity itself.

Ex. 9. Multiply $\frac{2x-5}{4}$ by 8.

Here $2x-5$ is to be divided by 4 and multiplied by 8; this is equivalent to simply multiplying it by 2, and the product is $4x-10$.

Ex. 10. Multiply $\frac{2x-5}{16}$ by 80.

Here $\frac{80}{16}=5$, $\therefore$ the product required is 5 times $2x-5$, or $10x-25$.

Ex. 11. Multiply $\frac{a+b}{a}$ by $\frac{a-b}{b}$.

The product $=\frac{a+b}{a}\times\frac{a-b}{b}$; and $a+b$ multiplied by $a-b=a^2-b^2$, $\therefore$ the product $=\frac{a^2-b^2}{ab}$.

[*Exercises* K, 16...20, and 31...38, p. 49.]

41. *To divide one fraction by another fraction.*

Rule. *Invert that fraction which is the divisor, (that is, putting the numerator in the denominator's place and the denominator in the numerator's) and then multiply this fraction by the other according to the rule for multiplication.*

Thus $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$. To prove the rule,

Since the *Quotient* is always such a quantity as multiplied by the *Divisor* will produce the *Dividend*, therefore if the *dividend* can be put into two *factors*, one of which is the *divisor*, the other must be the *quotient*. Now $\frac{a}{b}$, the *dividend*, $=\frac{a\times cd}{b\times cd}=\frac{acd}{bcd}=\frac{cad}{dbc}=\frac{c}{d}\times\frac{ad}{bc}$, of which two factors $\frac{c}{d}$ is the *divisor*, therefore the *quotient* $=\frac{ad}{bc}$, the other factor, which proves the Rule.

Ex. 1. Divide $\frac{2}{x}$ by $\frac{3}{y}$.

$$\frac{2}{x}\div\frac{3}{y}=\frac{2}{x}\times\frac{y}{3}=\frac{2y}{3x}.$$

Ex. 2. Divide $\frac{ax}{by}$ by $\frac{a}{b}$.

$$\frac{ax}{by}\div\frac{a}{b}=\frac{ax}{by}\times\frac{b}{a}=\frac{abx}{aby}=\frac{x}{y}.\quad \text{(Art. 35)}$$

Ex. 3. Divide $\frac{2ab}{3xy}$ by $\frac{b}{x}$.

$$\frac{2ab}{3xy} \div \frac{b}{x} = \frac{2ab}{3xy} \times \frac{x}{b} = \frac{2abx}{3bxy} = \frac{2a \,.\, bx}{3y \,.\, bx} = \frac{2a}{3y}. \quad \text{(Art. 35)}$$

Ex. 4. Divide $\frac{2a^2b}{10x^2y^2}$ by $\frac{ab}{2xy}$.

$$\frac{2a^2b}{10x^2y^2} \div \frac{ab}{2xy} = \frac{2a^2b}{10x^2y^2} \times \frac{2xy}{ab} = \frac{2a \,.\, ab \,.\, 2xy}{5xy \,.\, 2xy \,.\, ab} = \frac{2a}{5xy}.$$

Ex. 5. Divide $\frac{a-x}{4}$ by $\frac{2}{a}$.

$$\frac{a-x}{4} \div \frac{2}{a} = \frac{a-x}{4} \times \frac{a}{2} = \frac{a^2-ax}{8}.$$

Ex. 6. Divide $\frac{a^2-x^2}{ax}$ by $\frac{a+x}{a}$.

$$\frac{a^2-x^2}{ax} \div \frac{a+x}{a} = \frac{a^2-x^2}{ax} \cdot \frac{a}{a+x} = \frac{a-x}{x} \cdot \frac{a+x}{a} \cdot \frac{a}{a+x} = \frac{a-x}{x}.$$

Ex. 7. Divide $\frac{1+x^2+2x}{3x}$ by $\frac{1+x}{2x}$.

$$\text{Quotient} = \frac{1+x^2+2x}{3x} \cdot \frac{2x}{1+x} = \frac{1+x}{3} \cdot \frac{1+x}{x} \cdot \frac{2x}{1+x} = \frac{1+x}{3} \times 2 = \frac{2+2x}{3}.$$

[*Exercises* K, 28...30, and 39...46.]

EXERCISES. K.

(1) Multiply $\frac{x}{2}$ by 3.

(2) .... $\frac{3x}{2}$ by 2.

(3) .... $\frac{5x}{4}$ by 2.

(4) .... $\frac{x}{3}$ by 6.

(5) .... $\frac{a-x}{2}$ by 4.

(6) .... $\frac{7x}{15}$ by 60.

(7) Multiply $\frac{2x}{21}$ by 84.

(8) .... $\frac{3x-5}{2}$ by 6.

(9) .... $\frac{12+9x}{16}$ by 80.

(10) .... $\frac{8-7x}{4\frac{1}{2}}$ by 9.

(11) .... $\frac{6x+13}{1\frac{1}{4}}$ by 15.

(12) .... $\frac{2x-1}{7\frac{1}{2}}$ by 15.

(13) Multiply $\dfrac{3x+4}{5\frac{1}{2}}$ by 11.

(14) ... $\dfrac{x-1\frac{2}{3}}{2\frac{1}{3}}$ by 7.

(15) ... $\dfrac{2\frac{1}{2}-\frac{1}{4}x}{2\frac{1}{2}}$ by 10.

(16) ... $\dfrac{3x}{2}$ by $\dfrac{1}{2}$.

(17) Multiply $\dfrac{3x}{2}$ by $\dfrac{2x}{3}$.

(18) ... $\dfrac{2-3x}{4}$ by $\dfrac{4}{5}$.

(19) ... $\dfrac{1}{2x}$ by $\dfrac{2}{x}$.

(20) ... $\dfrac{x}{y}+\dfrac{y}{x}$ by $xy-\dfrac{1}{xy}$.

(21) Divide $\dfrac{5x}{2}$ by 5.

(22) ... $\dfrac{3x}{4}$ by 5.

(23) ... $\dfrac{3x}{4}$ by 6.

(24) ... $\dfrac{21ax}{4y}$ by $7a$.

(25) ... $\dfrac{2mn}{p}$ by $2n$.

(26) Divide $\dfrac{2x-4xy}{y}$ by $2x$.

(27) ... $\dfrac{3a+6ab}{4}$ by $3a$.

(28) ... $\dfrac{5xy}{z}$ by $\dfrac{2x}{y}$.

(29) ... $\dfrac{2abc}{3d}$ by $-\dfrac{ac}{bd}$.

(30) ... $-\dfrac{a^2xy}{2bc}$ by $-\dfrac{ay}{4x}$.

Multiply

(31) $x+\dfrac{1}{x}$ by $x+\dfrac{1}{x}$.

(32) $\dfrac{x}{y}+x$ by $\dfrac{y}{x}+\dfrac{1}{x}$.

(33) $\dfrac{1}{1+x}+\dfrac{1}{1-x}$ by $\dfrac{1}{2}$.

(34) $1-\dfrac{2a}{1+a}$ by $1+\dfrac{2a}{1-a}$.

(35) $\dfrac{1}{2}+\dfrac{m-3}{2}$ by $\dfrac{1}{3}+\dfrac{m-2}{3}$.

(36) $\dfrac{a}{b}+\dfrac{1}{2}\cdot\dfrac{b}{a}$ by $\dfrac{b}{a}-\dfrac{1}{2}\cdot\dfrac{a}{b}$.

(37) $\dfrac{a^2-ax}{b}$ by $\dfrac{b^2}{a-x}$.

(38) $\dfrac{a^2+ax+x^2}{a^2-ax+x^2}$ by $\dfrac{a-x}{a+x}$.

Divide

(39) $2+\dfrac{1}{x}$ by $1-\dfrac{2}{x}$.

(40) $\dfrac{2-x}{y}$ by $\dfrac{x}{1-x}$.

(41) $\dfrac{b-3a}{2ab}$ by $\dfrac{2a-b}{4a}$.

(42) 1 by $1+\dfrac{x}{4-x}$.

Divide

(43) $\frac{1}{2}$ by $\frac{1}{2}-\frac{x}{2}$.

(44) $\frac{1}{1+x}$ by $1-\frac{1}{1+x}$.

(45) ab by $\frac{b^2}{a-x}$.

(46) $\frac{a^3-x^3}{a^3+x^3}$ by $\frac{a-x}{a+x}$.

BRACKETS.

42. When it is intended to express, that a quantity of *two or more terms or factors* is to be operated upon, *as a whole*, the quantity is often inclosed within "*Brackets*", such as ( ), { }, [ ], &c., having the sign or symbol of the operation immediately affixed to the Brackets, as it would be to a quantity represented by a single letter.

Thus $a+(b-c)$ means that $b-c$ is to be *added* to a,
$a-(b-c)$ .... $b-c$ ...... *subtracted* from a,
$a\times(b-c)$ .... $b-c$ ...... *multiplied* by a,
$(b-c)\div a$ .... $b-c$ ...... *divided* by a,
$(b-c)^2$ .... $b-c$ ...... *squared*.
$\sqrt{(b-c)}$ .....the *square root of* $b-c$ is to be taken.
$(ab)^2$ .... a times b ... *squared*.

And that *brackets* have a significant meaning will be easily seen by striking them out in a particular case, and observing the result. Thus, if to express $b-c$ taken a times we were to write $a\times b-c$, we could not fairly obtain any other product from this than $ab-c$, instead of the true product $ab-ac$. Again $b-c^2$ would not express the square of $b-c$, but only that the square of c is to be subtracted from b, which is quite another thing.

43. Sometimes, in the place of *Brackets*, a straight line is used, called a *Vinculum* (Latin for 'a bond, or tie'), drawn over the several terms or factors which are to be operated upon as a whole.

Thus $a-\overline{b-c}$ means the same as $a-(b-c)$,
$\sqrt{\overline{b-c}}$ .... $\sqrt{(b-c)}$,
$\overline{b-c}|^2$ .... $(b-c)^2$; and so on.

It must also be carefully borne in mind that the line which separates the *numerator* and *denominator* of a *fraction* serves as a *Vinculum* for *both*. Thus $\frac{b-c}{a}$ means the same

as $\overline{b-c}\div a$, or $(b-c)\div a$; and $\frac{a-b}{c-d}$ means the same as $\overline{a-b}\div\overline{c-d}$, or $(a-b)\div(c-d)$.

44. The learner often finds much difficulty in the use of *Brackets* and *Vincula;* all which may be avoided by constant attention to this one RULE :—

Never make a bracket or vinculum to disappear except when the operation indicated by the sign or symbol affixed to it has been performed.

Thus in $a+(b-c)$ the bracket is introduced simply to signify that *the whole quantity* $b-c$ is *to be added* to a, the sign preceding the bracket being +. When, therefore, this addition *has been performed*, the bracket is no longer of any use, and may be omitted.

Similarly in $a-(b-c)$, the sign before the bracket being −, signifying that $b-c$ is *to be subtracted* from a, when the subtraction *has been done*, the bracket is no longer of any use, and may be omitted.

In these two cases, however, the above rule admits of modification; for

I. In the first case, since, by Art. 16, the addition of $b-c$ to a is performed by merely writing the quantities in one line with their proper signs, thus, $a+b-c$, it appears that *where the bracket is used for purposes of* ADDITION, *that is, is preceded by the sign* +, *the bracket may be struck out as of no value in the result.*

II. In the second case, $a-(b-c)$, since by Art. 19, the subtraction of one quantity from another is performed by changing the sign of every term in the quantity to be subtracted, and then *adding* by the rules for Addition, instead of $b-c$ to be *subtracted* from a, we may put $-b+c$ to be *added* to a, which gives $a-b+c$, by Art. 16. Hence *in cases, where a bracket is preceded by the sign* −, *the bracket may be struck out, if every sign within the bracket be first changed, that is,* + *into* −, *and* − *into* +.

But in all cases where *brackets* or *vincula* are used for purposes of Multiplication, Division, Involution, Evolution, &c. the Multiplication, or Division, or whatsoever operation it may be, must be *actually performed* before the *Brackets* or *Vincula* can disappear.

It may be worth while to notice further, that a bracket or vinculum sometimes serves *two* purposes at the same time: for example in $a^2-(a-b)^2$, or $a^2-\overline{a-b}|^2$, the bracket or

vinculum not only expresses that $a-b$ is to be squared, but also that when squared the whole result is to be subtracted from a^2; *both* which purposes must be satisfied, before the bracket or vinculum is dispensed with.

Ex. 1. Simplify $a+(a-b)$.

$$\begin{aligned} a+(a-b) &= a+a-b, \text{ by I,} \\ &= 2a-b. \end{aligned}$$

Ex. 2. Simplify $a+b+(a-b)$.

$$\begin{aligned} a+b+(a-b) &= a+b+a-b, \text{ by I,} \\ &= 2a. \end{aligned}$$

Ex. 3. Simplify $a-(a-b)$.

$$\begin{aligned} a-(a-b) &= a-a+b, \text{ by II,} \\ &= b. \end{aligned}$$

Ex. 4. Simplify $a+b-(a-b)$.

$$\begin{aligned} a+b-(a-b) &= a+b-a+b, \text{ by II,} \\ &= 2b. \end{aligned}$$

Ex. 5. Simplify $ac-\overline{a-b}\,.\,c$.

$$\begin{aligned} ac-\overline{a-b}\,.\,c &= ac-\overline{ac-bc}, \\ &= ac-ac+bc, \text{ by II,} \\ &= bc. \end{aligned}$$

Ex. 6. Simplify $\frac{a}{b}-\frac{a-b}{b}$.

$$\begin{aligned} \frac{a}{b}-\frac{a-b}{b} &= \frac{a-\overline{a-b}}{b}, \text{ by Art. 37,} \\ &= \frac{a-a+b}{b}, \text{ by II,} \\ &= \frac{b}{b}, \\ &= 1. \end{aligned}$$

Ex. 7. Simplify $1+\frac{a+x}{a-x}$.

$$\begin{aligned} 1+\frac{a+x}{a-x} &= \frac{a-x}{a-x}+\frac{a+x}{a-x}, \\ &= \frac{a-x+\overline{a+x}}{a-x}, \text{ by Art. 36,} \\ &= \frac{a-x+a+x}{a-x}, \text{ by I,} \\ &= \frac{2a}{a-x}. \end{aligned}$$

Ex. 8. Simplify $1-\dfrac{a-x}{a+x}$.

$$1-\frac{a-x}{a+x}=\frac{a+x}{a+x}-\frac{a-x}{a+x},$$
$$=\frac{a+x-\overline{a-x}}{a+x}, \text{ by Art. 37,}$$
$$=\frac{a+x-a+x}{a+x}, \text{ by II,}$$
$$=\frac{2x}{a+x}.$$

Ex. 9. Multiply $a-\dfrac{a-b}{2}$ by 2.

$$2\times\left(a-\frac{a-b}{2}\right)=2a-2\times\frac{a-b}{2}, \begin{cases}\text{erasing the bracket, because the}\\ \text{multiplication is performed,}\end{cases}$$
$$=2a-\frac{a-b}{1},$$
$$=2a-\overline{a-b},$$
$$=2a-a+b, \text{ by II,}$$
$$=a+b.$$

Ex. 10. Multiply $\dfrac{x}{2}-\dfrac{x-6}{5}$ by 10.

$$\text{The Product}=10\times\frac{x}{2}-10\times\frac{x-6}{5}, \text{ Art. 22,}$$
$$=\frac{10x}{2}-\frac{10(x-6)}{5}, \text{ Art. 38,}$$
$$=5x-2(x-6),$$
$$=5x-(2x-12),$$
$$=5x-2x+12, \text{ by II,}$$
$$=3x+12.$$

Ex. 11. Simplify $(a+b)^2-(a-b)^2$.

$$(a+b)^2-(a-b)^2=(a^2+2ab+b^2)-(a^2-2ab+b^2),$$
$$=a^2+2ab+b^2-a^2+2ab-b^2, \text{ by I and II,}$$
$$=4ab.$$

N.B. If a numeral or letter immediately precedes the first limb of a bracket *without any sign* intervening, the sign $\times$, or the word '*times*', is understood, and extends to the *whole quantity* within the *bracket*. Thus $4(a+x)$ signifies 4 *times* the sum of a and x; $3(a+b-c)$ signifies 3 *times* the quantity which results from adding together a and b and subtracting c; $5\left(\frac{a}{b}+\frac{c}{d}\right)$ signifies 5 *times* the sum of the two *fractions* $\frac{a}{b}$ and $\frac{c}{d}$; and so on.

Similarly $(a+b)(c+d)$ signifies $\overline{a+b}$ *times* $\overline{c+d}$, that is, $c+d$ *multiplied by* $a+b$; and so on: the bracket and quantity inclosed within it being considered as a *single term* with respect to any *operation* to be performed upon it. Thus $(a+b)(c+d)$ is the same as xy, where $a+b=x$, and $c+d=y$.

EXERCISES. L.

Simplify each of the following quantities:

(1) $ab+a(c-b)$.

(2) $4(1-x)+3x$.

(3) $2(a+x)-2(a-x)$.

(4) $2(a+b)(a-b)$.

(5) $5(1-x)+(1+5x)\times 2$.

(6) $\frac{a-x}{2}-\overline{x-2a}$.

(7) $\frac{1}{2}(a+b)-\frac{1}{2}(a-b)$.

(8) $(a+7)x^2+(b-7)x^2$.

(9) $2-(-4+5x)$.

(10) $1-(1-\overline{1-x})$.

(11) $(6a-\overline{b+c})-(a-\overline{b-2c})$.

(12) $\frac{1}{2}(a-x)(2a+x)+\frac{1}{2}x(a+x)$.

(13) $(1+x)(1-x)(1+x^2)$.

(14) $2\left(x^2-\frac{1}{4}\right)\div(2x+1)+\frac{1}{2}$.

(15) $\frac{1}{2}\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{1}{2}\left(\frac{a}{b}-\frac{c}{d}\right)$.

(16) $\left\{\frac{a(a+b)+b^2}{a}\right\}\div\left\{\frac{b(a+b)+a^2}{b}\right\}$.

(17) $4\times\left\{\frac{3}{8(1-x)}+\frac{1}{8(1+x)}\right\}$.

(18) $\dfrac{2x(2x-a)}{(a-2x)^2} + \dfrac{a}{a-2x}$.

(19) $\dfrac{2}{3}x(x+1)\{x+2-\dfrac{1}{2}(2x+1)\}$.

(20) $\{1-\overline{1-x}|^2\}x(2+x)$.

SIMPLE EQUATIONS OF ONE UNKNOWN QUANTITY.

45. If we say that $2x+3x=5x$, or $2(a+x)=2a+2x$, or such like, where an *equality* is expressed betwixt two quantities which differ only *in form,* (an equality which admits of no more question, than $2+3=5$, or $2\times\{1+5\}=12$) —and which holds true for *any value whatever of* x, such an expression is called an "*Identity*". But when we say $x+4=6$, or $2(1+x)=14$, or such like, in these cases it is clear that *only certain values of* x will permit the expressed equality to be true, and these are called "*Equations*".

In "*Equations*" the question always is, what value of the letter or letters not already known will *verify* the expressed equality? The finding of such value is called "*Solving*" the "*Equation*".

Thus, in the Equation $x+4=6$, to find the value of x, that is, to *solve* the Equation, we see at once that x is the number which added to 4 makes 6, $\therefore x=2$, and can be no other number. Again, in $2(1+x)=14$, to find x, since twice $\overline{1+x}=14$, $1+x$ must be 7, and $\therefore x=6$. But these 'Equations' are of a very simple character. It is obvious, that unknown quantities, one or more, *may be* involved in endless ways with others that are known, and form '*Equations*' of a much more complex character; and to find the values of the unknown quantities in such cases forms a chief part of the business of Algebra.

To this end the following Rules are needed. They all rest upon the axiom, or self-evident truth, that, if two quantities are equal to each other, and the same operation precisely be performed upon both, the results will still be equal.

46. Rule I. *If the same quantity be found on both sides the symbol* $=$, *and having the same sign,* $+$ *or* $-$, *it may be erased along with its sign from both sides.*

This is evident, for if equal quantities be taken from equal quantities the remainders are manifestly equal. Thus, if $x+4=7+4$, $+4$ may be struck out on both sides the symbol $=$, and leave $x=7$.

47. RULE II. *In an equation any term may be transposed from one side of the symbol = to the other, if its sign be changed, + to –, or – to +, as the case may be.*

For, let $ax+b=cx+d$ be the equation; then if from the equal quantities we subtract the same quantity, cx, the remainders will be equal; that is,

$$ax-cx+b=cx-cx+d,$$
$$\therefore\ ax-cx+b=d,\quad \because\ cx-cx=0;$$

thus cx has been *transposed* from one side of the symbol = to the other by changing its *sign*.

Again, subtract b from each side, then

$$ax-cx+b-b=d-b,$$
$$\text{or } ax-cx=d-b,\quad \because\ b-b=0,$$

that is, b has been *transposed* from one side to the other by changing its *sign*.

Ex. 1. Transpose the letters to one side, and the numbers to the other, in the equation $x+2=6-x$.

Here $x+x=6-2$, $-x$ being changed into $+x$, and $+2$, into -2, according to the rule.

Ex. 2. Transpose the literal terms to one side, and the numbers to the other, in the equation

$$4x-6=3x-2x+12.$$
$$\text{Here } 4x-3x+2x=12+6.$$

48. RULE III. *If every term of an equation be multiplied by the same quantity, the equality will still remain.*

For, in multiplying every *term* by the same quantity, the *whole* quantities which are equal (Art. 22) are equally multiplied, and therefore the products must be equal.

This rule is chiefly used for *clearing* an equation *of fractions*, if they stand in the way. Thus, taking the equation $7x-6=\frac{5x}{3}$, multiplying every term by 3, the denominator of the fractional term, we have

$$21x-18=5x,\ (\because\ 3\times\frac{5x}{3}=5x,)$$

in which no *fraction* appears.

Again, if the equation be $\frac{x}{2}+5=\frac{x}{3}+6$, multiplying by 2, we get $x+10=\frac{2x}{3}+12$, making *one* of the *fractions* to disappear. Next, multiplying by 3, the other denominator, we get

$3x+30=2x+36$, in which *no fractions* appear.

And the same thing may be done in like manner, whatever be the number of fractional terms.

But the readiest method of clearing an equation of several *fractional* terms is to make *one* multiplication serve for all, which may always be conveniently done, if the *denominators* of the fractions are not very large. Thus, in the Ex. before given,

$$\frac{x}{2}+5=\frac{x}{3}+6,$$

instead of multiplying first by 2, and then by 3, multiply at once by 2×3, or 6; then we get, *at one step*,

$$3x+30=2x+36, \because 6\times\frac{x}{2}=3x, \text{ and } 6\times\frac{x}{3}=2x.$$

Again, if the equation be $\frac{x}{2}-\frac{2x}{3}+\frac{x}{5}=6$, multiplying *at once* by $2\times 3\times 5$, or 30, we get

$$(\because 30\times\frac{x}{2}=15x,\ 30\times\frac{2x}{3}=20x,\ 30\times\frac{x}{5}=6x),$$

$$15x-20x+6x=180,$$

in which no *fractions* appear.

But sometimes instead of multiplying by the product of the *different* denominators it will save trouble to multiply by their "*Least Common Multiple*", that is, the least number which contains each of them without remainder. Thus, taking the equation

$$\frac{x}{2}-\frac{x}{4}+\frac{x}{8}=3,$$

in this case the product of the denominators is 64, but the small number 8 is their *Least Common Multiple*, and will serve as a multiplier to clear away the fractions. Thus, multiplying by 8, we get

$$(\because 8\times\frac{x}{2}=4x,\ 8\times\frac{x}{4}=2x,\ 8\times\frac{x}{8}=x),$$

$$4x-2x+x=24,$$

in which no *fractions* appear.

49. RULE IV. *If every term of an equation be divided by the same quantity, the equality will still remain.*

For, in dividing every *term* by the same quantity, the *whole* quantities which are equal (Art. 26) are equally divided, and therefore the quotients must be equal. Thus, if $4x-2x=16$, dividing every term by 2, (which will be taking the *half* of *equal* quantities) we get $2x-x=8$. Or, if $7x=28$ be the equation, $\frac{7x}{7}=\frac{28}{7}$, or $x=4$.

Again, if $ax=b$, $\frac{ax}{a}=\frac{b}{a}$, or $x=\frac{b}{a}$.

Having established these four Rules, we are now enabled to proceed with the *solution* of equations, at least of the simplest class of them, called "*Simple Equations of one Unknown Quantity*", meaning such equations as contain only one letter not already known, and that too in its simple 1st 'power', as x, not containing any higher power as x^2, x^3, &c. *after the preceding Rules have been applied to simplify it.*

50. *To solve a simple equation of one unknown quantity.*

(1) *Clear* the equation of fractions by Rule III., *if the unknown quantity be found in any,* (not otherwise).

(2) If any brackets or vincula remain, get rid of them by Art. 44.

(3) *Transpose* all the terms which contain the unknown quantity to one side of the symbol =, and those which do not to the other side, by Rule II.

(4) *Combine* similar quantities as much as possible by the rules of *Addition* and *Subtraction,* which will make the unknown quantity appear in *one term* only.

(5) *Divide* the whole equation by the coefficient of that term, and the required value of the unknown quantity will be found.

N.B. At *any stage* of the process of solution, as well as at the beginning, Rules I. and IV. must be brought to bear, when the equation will suffer it.

Ex. 1. If $3x+4-6=2x+7$, find the value of x.

Here the unknown x does not appear in a fractional form, nor are there any *brackets* or *vincula,* therefore we begin at once by

transposing, $3x-2x=6+7-4$,

and combining, $1x$, or $x=9$, the value required.

That this value of x is correct will appear by putting 9 for x in the given equation; for then we have

$$3\times9+4-6=2\times9+7, \text{ or } 25=25.$$

Ex. 2. If $7x-5x+\frac{1}{2}=\frac{3}{2}-3x+19$, find x.

Here the unknown x does not appear in a fractional form, therefore

transposing, $7x-5x+3x=\frac{3}{2}-\frac{1}{2}+19$,

combining, $5x=1+19=20$, $(\because \frac{3}{2}-\frac{1}{2}=1)$,

dividing by 5, $x=\frac{20}{5}=4$.

[*Exercises* M, 1...14, p. 61.]

Ex. 3. If $2x-3=\frac{x}{2}+6$, find the value of x.

Here the unknown x appears in a fractional form, $\frac{x}{2}$, therefore to clear the equation of this fraction, multiply the whole by the denominator 2, and we have

$$4x-6=x+12, \because 2\times\frac{x}{2}=x,$$

transposing, $4x-x=12+6$, (12 is the same here as $+12$),
combining, $3x=18$,
dividing by 3, $x=\frac{18}{3}=6$, the value required.

To shew that this value is correct, substitute it for x in the given equation: then we have

$$2\times6-3=\frac{6}{2}+6, \text{ or } 12-3=3+6, \text{ or } 9=9,$$

which is obviously true, shewing that the proposed equation is true, if $x=6$.

[*Exercises* M, 15...19, p. 61.]

Ex. 4. If $\frac{x}{2}-5=\frac{x}{3}-3$, find x.

Here the unknown quantity appears in *two* fractions, $\frac{x}{2}$ and $\frac{x}{3}$, therefore to clear the equation of both, multiply by 2×3, or 6, by Rule III., and we have

$$3x-30=2x-18, \because 6\times\frac{x}{2}=3x, \text{ and } 6\times\frac{x}{3}=2x,$$

transposing, $3x - 2x = 30 - 18$,
combining, $x = 12$.

This value of x is correct, for $\frac{12}{2} - 5 = 6 - 5 = 1$, and $\frac{12}{3} - 3 = 4 - 3 = 1$.

Ex. 5. If $\frac{x-6}{2} + 6 = \frac{5x-6}{2}$, find x.

Multiply by 2, $x - 6 + 12 = 5x - 6$,
erasing -6, $x + 12 = 5x$,
transposing, $12 = 5x - x$,
combining, $12 = 4x$,
dividing by 4, $3 = x$; or $x = 3$.

Ex. 6. If $\frac{x}{2} - \frac{5x}{3} - \frac{4}{3} = \frac{4x}{3} - 3$, find x.

Multiply by 2×3, or 6,

$3x - 10x - 8 = 8x - 18$,
transposing, $3x - 10x - 8x = 8 - 18$,
combining, $-15x = -10$,
dividing by -15, $x = \frac{-10}{-15} = \frac{2}{3}$.

[*Exercises* M, 20...24, p. 61.]

Ex. 7. If $\frac{x}{3} - \frac{x}{2} + \frac{x}{5} = \frac{1}{2}$, find x.

Multiply by $2 \times 3 \times 5$, or 30,

$(\because 30 \times \frac{x}{3} = 10x,\ 30 \times \frac{x}{2} = 15x,\ 30 \times \frac{x}{5} = 6x)$,

$10x - 15x + 6x = 15$,
combining, $x = 15$.

Ex. 8. If $\frac{4x}{3} - \frac{2x}{10} + \frac{x}{6} = 39$, find x.

The 'Least Common Multiple' of 3, 10, and 6, is 30, therefore to clear off fractions multiply by 30, and we have

$(\because 30 \times \frac{4x}{3} = 10 \times 4x = 40x$, &c.)

$40x - 6x + 5x = 1170$,
combining, $39x = 1170$,
dividing by 39, $x = \frac{1170}{39} = 30$.

[*Exercises* M, 25...36.]

EXERCISES. M.

Find the value of x in each of the following equations:

(1) $6x-10=5x-4$.

(2) $13x+1=9x+5$.

(3) $3x+30=2x+36$.

(4) $4x-2x=24-x$.

(5) $7x-11+5=8x-9$.

(6) $15-2x+6=3x+1$.

(7) $3x-6=12-4x-4$.

(8) $12-8x=15-3x-8$.

(9) $121=14x+1-3x+10$.

(10) $500=30x+12+32x-8$.

(11) $7x-2x+5=13x-4x-15$.

(12) $12x-6x+4x=3x+84$.

(13) $2x+\frac{1}{2}=3x-\frac{1}{2}$.

(14) $15x-3\frac{1}{2}=3\frac{1}{2}+x$.

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(15) $x+\frac{x}{2}=6$.

(16) $2x-\frac{x}{2}=18$.

(17) $3x+\frac{x}{3}=4x-6$.

(18) $\frac{4x}{3}+\frac{2}{3}=x+3$.

(19) $\frac{3x}{5}-\frac{x}{5}=x-6$.

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(20) $\frac{x}{3}+\frac{x}{6}=15$.

(21) $\frac{x}{5}-\frac{x}{10}=\frac{1}{2}$.

(22) $x-\frac{x}{2}+\frac{x}{3}-\frac{2}{3}=3\frac{1}{2}$.

(23) $\frac{2x}{7}+\frac{x}{6}-\frac{1}{6}=x-4$.

(24) $\frac{3x}{7}-1=\frac{x}{5}+\frac{3}{5}$.

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(25) $\frac{x}{2}-\frac{x}{3}-\frac{x}{4}+\frac{4}{3}=\frac{3}{4}$.

(26) $\frac{3x}{2}-\frac{2x}{3}+\frac{1}{2}=\frac{x}{6}+9\frac{5}{6}$.

(27) $\frac{x}{5}+\frac{x}{4}+\frac{x}{3}-\frac{x}{2}=17$.

(28) $x-\frac{x}{6}-\frac{x}{12}-\frac{x}{7}=\frac{x}{2}+9$.

(29) $\frac{3x}{14}-\frac{2x}{21}+\frac{1}{3}=\frac{x}{4}-4\frac{1}{4}$.

(30) $\frac{3x}{7}-\frac{x}{4}-\frac{x}{6}=\frac{5}{21}-\frac{x}{28}$.

(31) $2x-\frac{2x}{5}-2\frac{1}{5}-\frac{4x}{11}=\frac{8x}{7}-1\frac{6}{11}$.

(32) $\frac{x}{8}+\frac{2x}{5}=\frac{7x}{15}-\frac{x}{60}+\frac{3}{20}$.

(33) $\frac{7x}{8}-\frac{3x}{7}+1\frac{1}{8}=\frac{9x}{4}+\frac{9x}{14}-20\frac{25}{28}$.

(34) $\frac{3x}{16}+\frac{7x}{15}-\frac{7x}{20}=2\frac{19}{60}-\frac{3}{16}$.

(35) $\frac{14x}{3}-\frac{8x}{5}=10\frac{1}{3}+\frac{2x}{1\frac{1}{2}}-3\frac{2}{5}$.

(36) $\frac{x}{4}-4\frac{1}{2}+\frac{x}{5\frac{1}{2}}+\frac{x}{2}=\frac{16\frac{1}{4}}{5\frac{1}{2}}$.

51. When *brackets* or *vincula* appear in equations they are got rid of by the rules given in Art. 44.

Ex. 1. If $2(x+5)+3(2x-7)=21$, find x.

Since the 1st bracket here is used to shew that $x+5$ is to be taken *twice*, and the 2nd to shew that $2x-7$ is to be taken 3 times, and *added*, if we *perform* these multiplications we may then strike out the brackets, (Art. 44), and we have,

$$\{\because 2(x+5)=2x+10, \text{ and } 3(2x-7)=6x-21,\}$$

$$2x+10+6x-21=21,$$

$$\text{transposing, } 2x+6x=21+21-10,$$

$$\text{combining, } 8x=32,$$

$$\text{dividing, } x=\frac{32}{8}=4.$$

Ex. 2. If $2(x+5)-3(2x-7)=15$, find x.

$\because 2(x+5)=2x+10$, and $3(2x-7)=6x-21$, we have

$$2x+10-(6x-21)=15,$$

$$\text{or } 2x+10-6x+21=15, \text{ by Art. 44,}$$

$$\text{transposing, } 2x-6x=15-10-21,$$

$$\text{combining, } -4x=-16,$$

$$\text{dividing, } x=\frac{-16}{-4}=4.$$

Ex. 3. If $5-\dfrac{x+4}{11}=x-3$, find x.

Multiplying by 11, and observing that the line which separates the numerator and denominator of a fraction always serves as a *vinculum* to both, we have

$$55-\overline{x+4}=11x-33,$$

$$\text{or } 55-x-4=11x-33, \text{ by Art. 44,}$$

$$\text{transposing, } 55-4+33=11x+x,$$

$$\text{combining, } 84=12x,$$

$$\text{dividing, } x=\frac{84}{12}=7.$$

Ex. 4. If $x + \dfrac{3x-5}{2} = 12 - \dfrac{2x-4}{3}$, find x.

To clear off fractions multiply by 2×3, or 6, and we have

$$6x + 3(3x-5) = 72 - 2(2x-4),$$

$$\text{or } 6x + (9x-15) = 72 - (4x-8),$$

$$\therefore\ 6x + 9x - 15 = 72 - 4x + 8, \text{ by Art. 44},$$

transposing, $6x + 9x + 4x = 72 + 8 + 15$,

combining, $19x = 95$,

dividing, $x = \dfrac{95}{19} = 5$.

Ex. 5. If $\dfrac{8-7x}{8} + \dfrac{12+9x}{16} = \dfrac{1-3x}{10} - \dfrac{29+8x}{20}$, find x.

The Least Common Multiple of the denominators is 80, and multiplying by 80, we have

$$10(8-7x) + 5(12+9x) = 8(1-3x) - 4(29+8x),$$

$$\text{or}\quad 80 - 70x + 60 + 45x = 8 - 24x - 116 - 32x,$$

transposing, $24x + 32x - 70x + 45x = 8 - 116 - 60 - 80$,

combining, $31x = -248$,

dividing, $x = \dfrac{-248}{31} = -8$.

Ex. 6. If $\dfrac{1}{14}\left(3x + \dfrac{2}{3}\right) - \dfrac{1}{7}(4x - 6\tfrac{2}{3}) = \dfrac{1}{2}(5x-6)$, find x.

Multiply by 14, and we have

$$3x + \frac{2}{3} - 2(4x - 6\tfrac{2}{3}) = 7(5x-6),$$

$$\text{or } 3x + \frac{2}{3} - (8x - 12\tfrac{4}{3}) = 35x - 42,$$

$$\therefore\ 3x + \frac{2}{3} - 8x + 12\tfrac{4}{3} = 35x - 42,$$

transposing, $42 + \dfrac{2}{3} + 12\tfrac{4}{3} = 35x + 8x - 3x$,

combining, $56 = 40x$, $\because\ \dfrac{4}{3} + \dfrac{2}{3} = \dfrac{6}{3} = 2$,

dividing, $x = \dfrac{56}{40} = 1\tfrac{2}{5}$.

EXERCISES. N.

Find the value of x in each of the following equations:

(1) $6x+2(11-x)=3(19-x)$.

(2) $3(x+1)+2(x+2)=32$.

(3) $3x-2(5x+4)=2(4x-9)$.

(4) $5(2x-2)-3(2x+1)=27$.

(5) $6(3-2x)=24-4(4x-5)$.

(6) $45-4(x-2)=5(x+2)$.

(7) $7x=8-\frac{1-9x}{2}$.

(8) $\frac{2x}{7}+4=x-\frac{x-1}{6}$.

(9) $\frac{3x+1}{2}-\frac{x-1}{6}=\frac{2x}{3}+10$.

(10) $\frac{1}{4}(x+6)-\frac{1}{12}(16-3x)=4\frac{1}{6}$.

(11) $\frac{1}{16}(3x+3)+\frac{1}{15}(7x-4)-\frac{1}{20}(7x+1)=2$.

(12) $10\left(x+\frac{1}{2}\right)-6x\left(\frac{1}{x}-\frac{1}{3}\right)=23$.

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52. It will often happen that the unknown quantity is found in the *denominator* of one or more of the fractions; but the preceding methods are not materially affected thereby.

1st. If the denominators which contain the unknown quantity be *single terms*, no other method of solution is required but such as have been already applied. Thus,

Ex. 1. If $\frac{9}{2x}-4=5$, find x.

transposing, $\frac{9}{2x}=5+4$,

combining, $\frac{9}{2x}=9$,

multiply by $2x$, $9=18x$,

dividing, $x=\frac{9}{18}=\frac{1}{2}$.

Ex. 2. If $\frac{2}{x}+\frac{4}{x}=\frac{3}{x}+\frac{5}{x}-\frac{2}{17}$, find x.

Since the first 4 fractions have a Common Denominator,

by Addition, $\frac{6}{x}=\frac{8}{x}-\frac{2}{17}$.

transposing, $\frac{8}{x} - \frac{6}{x} = \frac{2}{17}$,

combining, $\frac{2}{x} = \frac{2}{17}$,

$\therefore\ x = 17.$

Ex. 3. If $\frac{3}{x} - \frac{2}{3x} = \frac{5}{3x} + \frac{1}{3}$, find x.

Multiply by $3x$, $9 - 2 = 5 + x$,
transposing, $x = 9 - 2 - 5$,
combining, $x = 2$.

[*Exercises* O, 1...4, p. 67.]

2nd. If any of the denominators which contain the unknown quantity consists of *two or more terms*, it will generally be advisable to clear the equation of the *simplest* denominators *first*, leaving the others to be dealt with afterwards, when, by *erasing*, *transposing*, and *combining*, the equation has been reduced to *fewer terms*. Or, if there be no *simple* denominators, then the *complex* denominators may be cleared off *singly one by one*, till all have disappeared.

Ex. 1. If $\frac{6x+13}{15} - \frac{3x+5}{5x-25} = \frac{2x}{5}$, find x.

Multiply by 15 to clear away the *simple* denominators first, and we have

$$6x + 13 - \frac{15(3x+5)}{5x-25} = 6x, \quad \because 15 \times \frac{2x}{5} = 6x,$$

erasing, and transposing, $13 = \frac{15(3x+5)}{5x-25}$,

or $13 = \frac{3(3x+5)}{x-5}$, dividing numerator and denominator of the fraction by 5.

Multiply by $x - 5$, $13x - 65 = 9x + 15$,
transposing, $13x - 9x = 65 + 15$,
combining, $4x = 80$,
dividing, $x = \frac{80}{4} = 20$.

Ex. 2. If $\frac{10x+17}{18} - \frac{12x+2}{11x-8} = \frac{5x-4}{9}$, find x.

To clear off first the denominators 18, and 9, multiply the whole by 18, and we have

$$10x + 17 - \frac{216x + 36}{11x - 8} = 10x - 8,$$

erasing, and transposing,

$$17 + 8 = \frac{216x + 36}{11x - 8},$$

combining, $$25 = \frac{216x + 36}{11x - 8},$$

multiply by $11x - 8$, $$25(11x - 8) = 216x + 36,$$

$$275x - 200 = 216x + 36,$$

transposing, $$275x - 216x = 200 + 36,$$

combining, $$59x = 236,$$

dividing, $$x = \frac{236}{59} = 4.$$

[*Exercises* O, 5...7, p. 67.]

Ex. 3. If $\frac{1}{x-1} - \frac{2}{x+7} = \frac{1}{7(x-1)}$, find x.

Multiply by $7(x-1)$, and we have

$$7 - \frac{14(x-1)}{x+7} = 1, \quad \because 7(x-1) \times \frac{1}{x-1} = 7,$$

transposing, and combining, $$6 = \frac{14(x-1)}{x+7},$$

multiply by $x + 7$, $$6x + 42 = 14x - 14,$$

transposing, $$14x - 6x = 42 + 14,$$

combining, $$8x = 56,$$

dividing, $$x = \frac{56}{8} = 7.$$

Ex. 4. If $\frac{2(3-4x)}{3-x} + \frac{3}{1-x} = 8$, find x.

Multiply by $3 - x$, and we have

$$2(3-4x) + \frac{9-3x}{1-x} = 24 - 8x,$$

$$\text{or } 6 - 8x + \frac{9-3x}{1-x} = 24 - 8x,$$

erasing, and transposing, $$\frac{9-3x}{1-x} = 24 - 6 = 18,$$

multiply by $1-x$, $\quad 9-3x=18-18x,$

transposing, $\quad 18x-3x=18-9,$

combining, $\quad 15x=9,$

dividing, $\quad x=\frac{9}{15}=\frac{3}{5}.$

Ex. 5. If $\frac{15+3x}{x+1}+\frac{30+4x}{x+3}=7+\frac{24}{x+1}$, find x.

Multiply by $x+1$, and we have

$$15+3x+\frac{30x+4x^2+30+4x}{x+3}=7x+7+24,$$

transposing, and combining,

$$\frac{34x+4x^2+30}{x+3}=4x+16,$$

multiply by $x+3$,

$$34x+4x^2+30=4x^2+16x+12x+48,$$

erasing, and transposing,

$$34x-16x-12x=48-30,$$

combining, $\quad 6x=18,$

dividing, $\quad x=\frac{18}{6}=3.$

[*Exercises* O, 8...11.]

EXERCISES. O.

Find the value of x in each of the following equations:

(1) $\frac{6}{x}-\frac{4}{x}+1=\frac{5}{x}+\frac{1}{4}.$

(2) $\frac{2}{3x}+\frac{3}{2x}=13.$

(3) $\frac{4}{5x}+\frac{5}{4x}=41.$

(4) $\frac{a}{bx}+\frac{b}{ax}=a^2+b^2.$

(5) $\frac{6x-4}{21}+\frac{x-2}{5x-6}=\frac{2x}{7}.$

(6) $\frac{9x-16}{36}=\frac{12-4x}{4-5x}+\frac{x-4}{4}.$

(7) $\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}.$

(8) $\frac{x-7}{x+7}+\frac{1}{2(x+7)}=\frac{2x-15}{2x-6}.$

(9) $\frac{3}{x}-\frac{2}{x+1}=\frac{5}{4(x+1)}.$

(10) $\dfrac{17}{6x+17} - \dfrac{10}{3x-10} = \dfrac{1}{1-2x}$.

(11) $\dfrac{6x+8}{2x+1} - \dfrac{2x+38}{x+12} - 1 = 0$.

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53. In cases where the *numerical* quantities are many and large, the following method of *writing* the solution of an equation will be found useful and convenient to young pupils, who are unaccustomed to add or subtract without having the quantities placed beneath each other.

Ex. 1. If $70x-42x+42+280=700-35x-60x+80+56x-56$, find x.

$$\text{Transposing, } \begin{array}{r|r|r|r|} 70 & x-42 & x=700 & -\ 56 \\ 35 & -56 & 80 & -\ 42 \\ 60 & & & -280 \end{array}\,,$$

$$\begin{array}{r|r|} 165 & x=\ \ 780 \\ -98 & -378 \end{array}\,,$$

$$67x = 402,$$

$$\therefore\ x = \frac{402}{67} = 6.$$

Ex. 2. If $\dfrac{9x-13}{4} - \dfrac{249-9x}{14} = \dfrac{7x+9}{8} - \dfrac{3x+1}{7}$, find x.

Here 56 is the Least Common Multiple of the Denominators, therefore multiply by 56, and we have

$$126x - 182 - \overline{996 - 36x} = 49x + 63 - \overline{24x + 8},$$

$$\text{or}\quad 126x - 182 - 996 + 36x = 49x + 63 - 24x - 8,$$

$$\text{transposing, } \begin{array}{r|r|l} 126 & x - 49x = \ \ 63 & -8, \\ 36 & 182 & \\ 24 & 996 & \end{array}$$

$$\text{combining, } \begin{array}{r|r|} 186 & x = 1241 \\ -49 & -8 \end{array}\,,$$

$$137x = 1233,$$

$$\therefore\ x = \frac{1233}{137} = 9.$$

Ex. 3. If $201(x-1) + 25(3x+1) + 22(5x+1) = 45(x+10) + 21(x+11) - 35$, find x. Ans. $x = 2\frac{1}{2}$.

PROBLEMS

DEPENDING UPON THE SOLUTION OF SIMPLE EQUATIONS OF ONE UNKNOWN QUANTITY.

54. We are now come to the application of all that has gone before to the solution of questions and problems, some of which might be solved by *Arithmetic*, but not so certainly and expeditiously, while others lie entirely beyond its reach. At this stage scarcely any rules can be laid down, which will be of much use to the learner. Practice only can make him quick and expert in bringing a proposed problem into the form of an *Equation:* and when that is once done, the solution of the *Equation* by the foregoing rules is the solution of the *Problem.*

It may be worth while, however, to urge the necessity of fully comprehending the *meaning* of every part of the problem proposed, as the first thing to be done. It should be seen clearly both what is *known* or *given*, and what is *unknown* and *required.* Then representing the latter by x, the student will be able, by a little practice, to express the conditions of the problem in terms composed of x and the *known* quantities, and at length to translate the whole into an *Equation.*

Prob. 1. The ages of 3 children together amount to 24 years, and they were born two years apart; what is the age of each?

Here we have

Known quantities.	*Unknown and required.*
1. The sum of the ages of all three, 24 years.	1. Age of youngest.
2. The difference between the ages of any two of them.	2. Age of next.
	3. Age of the oldest.

But, in reality, we have only *one* unknown quantity to *find,* because when we know the age of one of the children, the ages of the two others immediately follow. So that, we say,

let x be the age of the youngest,

then $x + 2 =$ .... next,

and $x + 4 =$ .... oldest.

Thus far we have algebraized *one* of the two known conditions of the problem. There still remains to notice, that

the sum of the ages is 24 years. Now this sum is $3x+6$, adding together x, $x+2$, and $x+4$;

$\therefore\ 3x+6=24$, an *equation* from which to find x.

Transposing, $3x=24-6$, or 18,

dividing, $x=\frac{18}{3}$, or 6.

$\therefore$ the age of the youngest is 6 years,

.... next ... 8 ......

.... oldest ... 10 ......

Prob. 2. I have exactly 5 times as many shillings as sovereigns, and altogether my money amounts to £8. 15*s*. How many have I of each?

Let x be the number of sovereigns,

then $5x=$ .... shillings.

Now x sovereigns $=x$ times 20 shillings $=20x$ shillings,

$\therefore\ 20x+5x$, or $25x$, $=$ all the money, in shillings.

But £8. 15*s*. $=175$ shillings,

$\therefore\ 25x=175$,

dividing, $x=\frac{175}{25}=7$, the number of sovereigns,

and $5x=5\times7=35$, the number of shillings.

Prob. 3. I went to the bank with a cheque for 6 guineas and asked to have for it exactly *the same number* of sovereigns, half-sovereigns, shillings, and sixpences. The banker was puzzled: what is the number?

Let x be the number required,

then x sovereigns contain x times 20, or $20x$, shillings,

x half-sovereigns ...... x times 10, or $10x$, shillings,

x shillings ...... x shillings,

x sixpences ...... $\frac{x}{2}$ shillings,

6 guineas ...... 6 times 21, or 126, shillings.

Therefore, by the question,

$$20x+10x+x+\frac{x}{2}=126,$$

$$\text{or }31x+\frac{x}{2}=126,$$

$$\text{multiply by 2, } 62x+x=252,$$

$$63x=252,$$

$$\therefore\ x = \frac{252}{63} = 4, \text{ the number required.}$$

		£.	s.	d.
Verification.	4 sovereigns,	4	0	0
	4 half-sovereigns,	2	0	0
	4 shillings	0	4	0
	4 sixpences	0	2	0
	Total	£6	6	0

PROB. 4. What is the number of which the 3rd and 6th parts together make 15?

Let x be the number required,

$$\left.\begin{array}{l} \text{then } \frac{x}{3} = \text{its 3rd part,} \\ \text{and } \frac{x}{6} = \text{its 6th } \ldots\ldots \end{array}\right\} \text{the sum of which parts, by the question, is equal to 15,}$$

$$\text{that is, } \frac{x}{3} + \frac{x}{6} = 15,$$

$$\text{multiply by 6,} \quad 2x + x = 90,$$

$$\text{combining,} \quad 3x = 90,$$

$$\therefore\ x = \frac{90}{3} = 30.$$

This value is correct, $\because \frac{30}{3} = 10,\ \frac{30}{6} = 5,$ and $10 + 5 = 15$.

PROB. 5. A certain garden contained three times as many gooseberry-trees as apple-trees. Afterwards four of each were cut down, and then there were *four* times as many gooseberry-trees as apple-trees. How many were there of each at first?

Let x be the number of apple-trees at first,
then $3x =$ .... gooseberry-trees at first.
Afterwards $x - 4 =$ number of apple-trees,
and $3x - 4 =$ .... gooseberry-trees,

therefore, by the question,

$$3x - 4 = 4(x - 4),$$

$$\text{or } 3x - 4 = 4x - 16,$$

$$\text{transposing, } 16 - 4 = 4x - 3x,$$

$\therefore\ x = 12$, the number of apple-trees at first,

and $3x = 36$, ... gooseberry-trees at first.

[*Exercises* P, 1...16, p. 81.]

Prob. 6. The date of the accession of Geo. III is represented by $1800-2x$, that of Geo. IV by $1800+\frac{1}{2}\times 2x$, that of Will. IV by $1800+\frac{1}{3}\times 3x$; and if Geo. IIIrd's reign be increased by $2x$, it will amount to 100 years. What are the actual dates?

The length of Geo. IIIrd's reign

$$\begin{aligned} &= 1800+\tfrac{1}{2}\times 2x-(1800-2x), \\ &= 1800+x-1800+2x, \\ &= 3x. \end{aligned}$$

$\therefore$ by the question, $3x+2x=100$,

$$\text{or, } 5x=100, \ \therefore x=\frac{100}{5}=20.$$

$\therefore$ accession of Geo. III is A.D. $1800-40$, or 1760.
.... Geo. IV ...... $1800+20$, or 1820.
.... Will. IV ...... $1800+30$, or 1830.

[Strictly speaking Geo. III did not reign full 60 years, having ascended the throne, *Oct.* 25, 1760, and died *Jan.* 29, 1820.]

Prob. 7. Divide 42 into 4 parts which shall be 4 consecutive numbers.

Let x be one part,

then $x+1$, $x+2$, $x+3$, are the other parts,

and $x+(x+1)+(x+2)+(x+3)=42$, by the question,

$$\text{combining, } 4x+6=42,$$
$$4x=36,$$

$$\therefore x=9; \text{ and } x+1=10, \ x+2=11, \ x+3=12;$$

$\therefore$ 9, 10, 11, 12, are the required parts.

Prob. 8. A man dies and leaves a widow, two sons, and three daughters; and in his will he orders that his personal property, amounting to £1700. shall be so divided, that the three daughters shall have as much as the two sons, and the widow as much as a son and a daughter together. What are their respective shares?

Let x be a son's share,

then $2x$ = the whole fortune of the 3 daughters,

$$\therefore \frac{2x}{3} = \text{a daughter's share,}$$

$$\text{and } x+\frac{2x}{3}, \text{ or } \frac{5x}{3} = \text{the widow's share;}$$

$$\text{hence } 2x + 2x + \frac{5x}{3} = 1700£.,$$

$$4x + \frac{5x}{3} = 1700,$$

$$\frac{17x}{3} = 1700,$$

$$\text{dividing by 17, } \frac{x}{3} = 100,$$

$$\therefore\ x = 300£., \text{ a son's share,}$$

$$\frac{2x}{3} = 200£., \text{ a daughter's share,}$$

$$\frac{5x}{3} = 500£., \text{ the widow's share.}$$

PROB. 9. A pump which lifts 2 gallons of water at each stroke, and makes 3 strokes in 2 minutes, is to be replaced by another which can make only 2 strokes in 3 minutes. What must be the discharge of the latter per stroke, to do the same work?

Let x be the number of gallons per stroke of the latter,

then $2x$ = gallons discharged in 3 minutes.

Now the 1st pump discharges 6 gallons in 2 minutes, which is at the rate of 3 gallons per minute; therefore in 3 minutes the discharge would be 9 gallons. Hence, by the question,

$$2x = 9,$$

$$\therefore\ x = \frac{9}{2} = 4\tfrac{1}{2}, \text{ the number of gallons required.}$$

PROB. 10. *A* and *B*, living on the same road $4\frac{1}{2}$ miles apart, set off from their homes at the same instant to meet each other, walking at the rate of 5 miles, and 4 miles, per hour respectively. Where will they meet? And how long after *B* might *A* set off, so as to meet exactly halfway?

1st. When they start together, let x be the number of miles *A* walks before they meet, then $4\frac{1}{2} - x = B$'s walk, in miles; and the *time* is the same for both.

$$\text{But } A\text{'s time} = \frac{\text{number of miles}}{\text{number per hour}} = \frac{x}{5},$$

$$\text{and } B\text{'s} \ \ldots = \ldots\ldots\ldots\ldots\ldots\ldots = \frac{4\frac{1}{2} - x}{4},$$

$$\therefore \text{ by the question, } \frac{x}{5} = \frac{4\frac{1}{2} - x}{4},$$

$$\text{multiply by } 4 \times 5, \text{ or } 20, \quad 4x = 22\tfrac{1}{2} - 5x,$$

$$9x = 22\tfrac{1}{2},$$

$$\therefore x = \frac{22\frac{1}{2}}{9} = 2\tfrac{1}{2}.$$

Therefore A and B meet $2\frac{1}{2}$ miles from A's home, and 2 miles from B's.

2nd. To meet exactly halfway; since A, walking at the rate of 5 miles per hour, would take $\frac{2\frac{1}{4}}{5}$ hours; and B $\frac{2\frac{1}{4}}{4}$ hours, walking 4 miles per hour; it is obvious that A might set off

$$\frac{2\frac{1}{4}}{4} - \frac{2\frac{1}{4}}{5} \text{ hours after } B,$$

$$\text{that is, } 2\tfrac{1}{4}\left(\frac{1}{4} - \frac{1}{5}\right), \text{ or } 2\tfrac{1}{4} \times \frac{1}{20}, \text{ hours,}$$

$$\ldots\ldots \frac{9}{80} \times 60\,min. \text{ or } 6\tfrac{3}{4}\,min.$$

Prob. 11. A miller has two kinds of wheat, one worth 7 shillings, and the other 6 shillings, per bushel; he wishes to make a mixture worth 6*s.* 8*d.* per bushel. How must he do it?

Let x be the number of bushels of the former, (either whole or fractional), which added to one of the other, will make the mixture required.

Then the value of these $(x + 1)$ bushels will be $(7x + 6)$ shillings. But, by the question, it must be $(x + 1)$ times 6*s.* 8*d.*, that is, $(x + 1) \times 6\frac{2}{3}$ shillings,

$$\therefore 7x + 6 = (x + 1) \times 6\tfrac{2}{3},$$

$$= 6x + \tfrac{2}{3}x + 6\tfrac{2}{3}, \quad \because 6\tfrac{2}{3} = 6 + \tfrac{2}{3},$$

$$7x - 6x - \tfrac{2}{3}x = 6\tfrac{2}{3} - 6,$$

$$\tfrac{1}{3}x = \tfrac{2}{3},$$

$$\therefore x = 2.$$

Therefore two bushels of the better sort added to one of the other will make the mixture required.

Prob. 12. A man and a boy undertake to dibble a field of beans, the man being able to do the whole work himself in 5 days, and the boy in 7 days. How long will it take them working together?

Let x be the number of days required;

now, because the man can do the whole in 5 days,

$$\text{the man's work per day} = \tfrac{1}{5} \text{ of the whole};$$

$$\text{similarly, the boy's} \; \ldots\ldots\ldots\ldots = \tfrac{1}{7} \text{ of the whole};$$

$$\therefore \text{ the man's and boy's together} = (\tfrac{1}{5} + \tfrac{1}{7}) \text{ of the whole}$$

$$= \tfrac{12}{35} \qquad \ldots\ldots\ldots$$

But the man and boy together can do the whole in x days, therefore the man and boy together can do in one day $\dfrac{1}{x}$ of the whole;

$$\therefore \frac{12}{35} = \frac{1}{x}, \qquad \text{or } x = \frac{35}{12} = 2\tfrac{11}{12} \text{ days.}$$

Prob. 13. Her Majesty, Queen Victoria, was born May 24, A.D. x, and Prince Albert was born Aug. 26, A.D. $(x+1)$. Now their united ages at the present time, Aug. 26, 1848, amount to *three* times the age of Prince Albert on the birth-day immediately preceding his marriage, which took place Feb. 10, 1840. What is the year of our Lord in which each was born?

Let x, and $x+1$, as stated in the question, be the years required, then, on the 26th Aug. 1848,

$$1848 - x = \text{age of the Queen,}$$

$$\text{and } 1848 - (x+1) = \ldots\ldots \text{ the Prince.}$$

Also, the age of the Prince on the birth-day preceding his marriage

$$= 1839 - (x+1),$$

therefore, by the question,

$$1848 - x + 1848 - (x+1) = 3\{1839 - (x+1)\},$$

$$\text{or, } 1848 - x + 1848 - x - 1 = 5517 - 3x - 3,$$

$$3x - 2x = 5517 - 3 + 1 - 1848 - 1848;$$

$$\therefore x = \left.\begin{matrix} 5518 \\ -3699 \end{matrix}\right\} = 1819, \text{ year of Queen's birth,}$$

$$\text{and } x + 1 = 1820, \ldots\ldots \text{ Prince's } \ldots$$

PROB. 14. A cask is filled by means of 3 cocks, which would fill it singly in 5, 6, and 10 minutes. In what time will the cask be filled, when they all run together?

Let x be the number of minutes required.

Now $\because$ one of the cocks will do the whole work in 5 minutes, its work *per min.* is $\frac{1}{5}$ of the whole. Similarly, the work *per min.* of each of the others is $\frac{1}{6}$, and $\frac{1}{10}$, of the whole. And therefore the work of all three together *per min.* $=\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{10}\right)$ of the whole work.

But x being the number of minutes in which all three will do the work, the work of the three *per min.* will be $\frac{1}{x}$ of the whole:

$$\therefore \frac{1}{5}+\frac{1}{6}+\frac{1}{10}=\frac{1}{x},$$

$$\frac{6+5+3}{30}, \text{ or } \frac{14}{30}=\frac{1}{x},$$

$$\therefore x=\frac{30}{14}=\frac{15}{7}=2\tfrac{1}{7} \text{ minutes.}$$

PROB. 15. A person, being asked what o'clock it was, replied that it was between one and two, and that the hour and minute hands were together. What was the time of day?

At one o'clock it is obvious that the hour and minute hands are separated from each other by exactly 5 of the minute divisions.

If then x be the number of minutes past one, at which the hour and minute hands are together, it is evident that in that time the minute hand has travelled over a space x, and the hour hand $x-5$ in the same time. But we know that the minute hand always goes 12 times as fast as the hour hand, and will therefore pass over 12 times the space in the same time.

$$\text{Hence } x=12(x-5),$$
$$=12x-60,$$

$$11x = 60,$$

$$\therefore\ x = \frac{60}{11} = 5\tfrac{5}{11};$$

that is, the hands are together at $5\frac{5}{11}$ min. past one.

PROB. 16. The distance from Manchester to Liverpool by rail is $31\frac{1}{2}$ miles; the express down-train leaves Manchester at 11.30 A.M. and arrives at Liverpool at 12.30; the up-train leaves Liverpool at 11.45 A.M. and arrives at Manchester at 12.35. Both trains perform the whole journey without stopping at any intermediate station; supposing the speed of each to be uniform, find where they will meet.

Let x be the number of miles from Manchester to the place of meeting,

then $31\frac{1}{2} - x =$ the number of miles from Liverpool to the place of meeting.

Now since the down-train travels $31\frac{1}{2}$ miles in 60 min., it travels the 60th part of that distance in 1 min., that is, $\frac{31\frac{1}{2}}{60}$ miles.

Similarly, the up-train travels $\frac{31\frac{1}{2}}{50}$ miles per min., therefore the number of minutes in which the down-train performs x miles $= \frac{\text{number of miles}}{\text{number per minute}} = x \div \frac{31\frac{1}{2}}{60}$, and the number of minutes in which the up-train performs $\overline{31\frac{1}{2} - x}$ miles $= (31\frac{1}{2} - x) \div \frac{31\frac{1}{2}}{50}$.

But the down-train starts 15 min. before the up-train;

$\therefore$ time of down-train for x miles $=$ time of up-train for $\overline{31\frac{1}{2} - x}$ miles $+ 15$,

$$\text{or } x \div \frac{31\frac{1}{2}}{60} = \overline{31\tfrac{1}{2} - x} \div \frac{31\frac{1}{2}}{50} + 15.$$

$$\frac{60x}{31\frac{1}{2}} = 50 - \frac{50x}{31\frac{1}{2}} + 15,$$

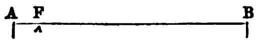
$$\frac{60 + 50}{31\frac{1}{2}} \cdot x = 65,$$

$$x = 31\tfrac{1}{2} \times \frac{65}{110} = 31\tfrac{1}{2} \times \frac{13}{22} = \frac{63 \times 13}{44} = 18\tfrac{27}{44};$$

therefore the trains will meet $18\frac{27}{44}$ miles from Manchester.

[*Exercises* P, 17...25, p. 82.]

Prob. 17. Supposing (as is proved in Treatises on Mechanics) that *the effect* of a power or weight, acting at right angles to a straight lever, *to turn it round its fulcrum*, is measured by that power or weight *multiplied* by its distance from the fulcrum, find where the fulcrum must be placed to enable a man to move a bale of goods weighing 552 lbs. by means of a lever 6 feet long, and exerting a power equal to 24 lbs. only.



Let AB represent the lever, the power being applied at B to raise the weight at A; and supposing F the fulcrum, let $AF = x$; then $BF = 6 - x$ feet; and, by the question,

$552 \times x =$ the *effect* of the Weight on one side of the fulcrum;

and $24(6-x) =$ the *effect* of the Power on the other side of the fulcrum.

When these two *effects*, then, are equal, the lever is exactly *balanced*, in which case

$$552x = 24(6-x),$$
$$= 144 - 24x,$$
$$\text{or } 576x = 144,$$
$$\therefore\ x = \frac{144}{576} = \overset{\text{ft.}}{\frac{1}{4}} = \overset{\text{in.}}{\frac{12}{4}} = 3\text{ in.}$$

Hence, with the fulcrum at 3 in. from A, the power and weight are exactly poised; and it is obvious that by moving the fulcrum nearer to A than 3 inches, which diminishes the *effect* of the weight and increases that of the power, the power will get the mastery, and the weight will be raised.

Prob. 18. If the '*Specific Gravity*' of pure milk be 1·03, and a certain mixture of milk and water be found, (by means of an instrument for the purpose) to be of *Specific Gravity* 1·02625, how much water has been added?

[Definition. By the '*Specific Gravity*' of a substance is meant the number of times which its weight is of an equal bulk of *water*. Thus the *Specific Gravity* of silver is 10·5, or $10\frac{1}{2}$, which means that any quantity of silver is $10\frac{1}{2}$ times the weight of the same quantity, *in bulk*, of *water*. The *Specific Gravity* of milk being 1·03 signifies that milk is $1\frac{3}{100}$ times as heavy as *water*; and so on.]

Let 1 quart of water be added to x quarts of pure milk to form the mixture; then,

$\because$ weight of x quarts of pure milk

$$= 1\cdot03 \text{ times weight of } x \text{ quarts of water,}$$
$$= 1\cdot03 \times x \times \text{weight of 1 quart of water,}$$

$\therefore$ whole weight of water and milk

$$= (1 + 1\cdot03 \times x) \times \text{weight of 1 quart of water.}$$

But there are $1 + x$ quarts of the mixture of specific gravity $1\cdot02625$,

$\therefore$ whole weight of this

$$= 1\cdot02625 \times \overline{1 + x} \times \text{weight of 1 quart of water,}$$
$$\therefore\ 1 + 1\cdot03 \times x = 1\cdot02625\,(1 + x),$$
$$(1\cdot03 - 1\cdot02625)\,x = 1\cdot02625 - 1,$$
$$\cdot00375x = \cdot02625,$$
$$\therefore\ x = \frac{\cdot02625}{\cdot00375} = 7.$$

Hence it appears, that one quart of water is added to 7 quarts of milk; consequently *one-eighth* of the mixture is water.

PROB. 19. A person observes the discharge of a gun at a distance, and hears the report exactly $10\frac{1}{2}$ seconds afterwards. Assuming that light travels at the rate of 192000 miles, and sound 1090 feet, per second, what is the distance between him and the gun?

Let x be the required distance in *miles*, then the number of seconds in which the *light* travels to the observer $= \frac{x}{192000}$; and $\because$ x miles $= 3 \times 1760 \times x$ feet, the number of seconds in which the *sound* travels to the observer $= \frac{3 \times 1760 \times x}{1090}$;

$$\therefore \text{ by the question, } \frac{3 \times 1760 \times x}{1090} - \frac{x}{192000} = 10\tfrac{1}{2},$$
$$\text{or } \frac{3 \times 176 \times 192000 - 109}{109 \times 192000} \cdot x = 10\tfrac{1}{2},$$
$$\therefore\ x = \frac{109 \times 192000 \times 10\frac{1}{2}}{3 \times 176 \times 192000 - 109},$$
$$= \frac{219744000}{101375891} = 2\tfrac{1}{6} \text{ miles nearly.}$$

PROB. 20. The *Specific Gravity* of gold is $19\frac{1}{4}$, and of silver $10\frac{1}{2}$: a goldsmith offers a mass of $\frac{1}{4}$ of a cubic foot, which he asserts to be gold, and which is found to weigh 260 lbs. 1st. Can it be all gold? 2nd. May it be adulterated with silver? 3rd. If the latter, what is the proportion of silver to gold? (*From De Morgan's Algebra.*)

[N. B. A Cubic foot of water weighs 1000 ounces avoirdupois.]

1. Since a cubic foot of water weighs 1000 oz. and gold is $19\frac{1}{4}$ times as heavy as water, a cubic foot of gold weighs $19\frac{1}{4}\times 1000$, or 19250 oz.; and $\frac{1}{4}$ of a cubic foot will be $4812\frac{1}{2}$ oz. or 300 lbs. $12\frac{1}{2}$ oz., *therefore the mass is not all gold.*

2. Since a cubic foot of silver weighs $10\frac{1}{2}\times 1000$, or 10500 oz., and $\frac{1}{4}$ of a cubic foot will be 2625 oz., or 164 lbs. 1 oz., therefore the mass is heavier than its bulk of silver, and lighter than its bulk of gold, and consequently may be a mixture of the two.

3. In this case, let $\frac{1}{x}$ of a cubic foot be the quantity of gold; then $\frac{1}{4}-\frac{1}{x}$ is the silver; and $\frac{19250}{x}$ is the weight of the gold, and $10500\left(\frac{1}{4}-\frac{1}{x}\right)$ the weight of the silver, in ounces. But the whole weight is 260 lbs., or 4160 oz.

$$\therefore \frac{19250}{x}+10500\left(\frac{1}{4}-\frac{1}{x}\right)=4160,$$

$$\text{or } 19250+2625x-10500=4160x,$$

$$4160x-2625x=19250-10500,$$

$$1535x=8750,$$

$$\therefore x=\frac{8750}{1535}=\frac{1750}{307}=\frac{175}{30}, \text{ or } \frac{35}{6}, \text{ nearly.}$$

$\therefore \frac{1}{x}=\frac{6}{35}$ nearly, the quantity of gold, in fractions of a cubic foot,

$$\frac{1}{4}-\frac{1}{x}=\frac{1}{4}-\frac{6}{35}=\frac{11}{140}, \text{ the quantity of silver......}$$

Hence, $\because \frac{6}{35}=\frac{24}{140}$, if a cubic foot be divided into 140 equal parts, in the proposed mass there are 24 such parts of gold, and 11 of silver.

EXERCISES. P.

1. What number is that which added to its half makes 24?

2. What number is that which increased by two-thirds of itself becomes 20?

3. What number is that of which the half exceeds the third part by 3?

4. What number is that of which the fourth part exceeds the fifth part by 3?

5. There is a certain number which, upon being diminished by 6, and the remainder multiplied by 6, produces the same result as if it were diminished by 4, and the remainder multiplied by 4. What is the number?

6. Divide 40 into two such parts, that one-tenth of the smaller part taken from one fifth of the greater will leave 5 for a remainder.

7. Divide 25 into two such parts that one shall be three-fourths of the other.

8. Find two numbers which produce the same result, 7, whether the one be subtracted from the other, or the latter be divided by the former.

9. Divide £1. among 4 children so that the oldest shall have 1*s.* more than the second, the second 1*s.* more than the third, and the third 1*s.* more than the youngest.

10. Divide a line 33 feet long into 4 parts, the second of which is $1\frac{1}{2}$ feet greater than the first, the third $2\frac{1}{2}$ feet greater than the second, and the fourth $3\frac{1}{2}$ feet greater than the third.

11. A banker was asked to pay £10. in sovereigns, and half-crowns, and so that the number of the latter should be exactly twice that of the former. How must he do it?

12. Thirteen shillings is the sum of exactly *the same number* of shillings, sixpences, pence, and halfpence. What is the number?

13. I have exactly 5 times as many shillings as half-crowns; and altogether my money amounts to £3. How many have I of each coin?

14. A father is 4 times as old as his son; but 3 years ago he was 7 times as old as the son. What is the age of each?

15. The ages of two brothers, who differ only by a single year, when added together amount to the age of their father; and if the father's age be increased by one-fourth of that of the elder brother, it will amount to four-score years. What is the age of each?

16. The ages of a man and his wife together amount to 80 years, and 20 years ago the woman was exactly two-thirds the age of the man. What is the age of each?

17. There is a certain fraction whose denominator is greater than its numerator by 1; and if 1 be taken from the numerator and added to the denominator, the fraction becomes equal to $\frac{1}{2}$. What is the fraction?

18. A certain fraction has its numerator less than its denominator by 2, and if 1 be taken from the numerator, and the numerator be added to the denominator to form a new denominator, the resulting fraction is equal to $\frac{1}{4}$. What is the fraction?

19. A boy being asked to divide one half of a certain number by 4, and the other half by 6, and to add together the quotients, attempted to obtain the required result *at one step* by dividing the whole number by 5; but his answer was too small by 2. What was the number?

20. Find the time between 12 and 1 o'clock when the hour and minute hands of a clock point exactly in opposite directions.

21. A person, being asked what o'clock it was, answered that it was between 5 and 6, and that the hour and minute hands were together. Required the time of day.

22. A servant is despatched on an errand to a town 8 miles off, and walks at the rate of 4 miles an hour: ten minutes afterwards another is sent to fetch him back, walking $4\frac{1}{2}$ miles per hour. How far from the town will the latter overtake the former?

23. A student has just an hour and a half for exercise. He starts off on a coach which travels 10 miles an hour, and after a time he dismounts, and walks home at the rate of 4 miles an hour. What is the greatest distance he can travel by the coach, so as to keep within his time?

24. A cistern which holds 820 gallons is filled in 20 minutes by 3 pipes, one of which conveys 10 gallons more, and another 5 gallons less, per minute, than the third. How much flows through each pipe per minute?

25. A man and a boy engaged to draw a field of turnips for 21*s.* but when two-fifths of the work was done, the boy ran away, and the man then finished it alone. The consequence was that the work occupied $1\frac{1}{4}$ days more than it should have done. Now the boy could do only half a man's work, and is paid in proportion. What did each receive per day?

SIMPLE EQUATIONS OF TWO UNKNOWN QUANTITIES.

55. If a *single* equation contain *two* unknown quantities, x and y, as $2x + 3y = 20$, then, transposing, $2x = 20 - 3y$, and, dividing, $x = 10 - \frac{3y}{2}$; but since this gives the value of one unknown quantity only in terms of the other, which is itself *unknown*, it furnishes no actual *solution* of the equation. Now, if besides $2x + 3y = 20$, there is given also another equation, as $3x + 2y = 25$, which holds true for the same values of x and y which belong to the former one, then from this we get

$$3x = 25 - 2y, \text{ and } x = \frac{25}{3} - \frac{2y}{3};$$

so that we have, from the *two* equations, $\because$ x has the same value in both by supposition,

$$10 - \frac{3y}{2} = \frac{25}{3} - \frac{2y}{3}, \text{ an equation of } one$$

unknown quantity,

multiply by 6, $\quad 60 - 9y = 50 - 4y,$

$$60 - 50 = 9y - 4y,$$

$$10 = 5y; \therefore y = \frac{10}{5} = 2.$$

Also, from 1st equation,

$$x = 10 - \frac{3y}{2} = 10 - \frac{6}{2} = 7, \quad (\because 3y = 6).$$

Hence the Solution of

$$\left.\begin{array}{r} 2x + 3y = 20, \\ \text{and } 3x + 2y = 25, \end{array}\right\}^* \text{ is } \left.\begin{array}{l} x = 7, \\ y = 2, \end{array}\right\} \begin{array}{l} \text{which upon trial is} \\ \text{found to verify.} \end{array}$$

This method of *eliminating*, as it is called, one of the unknown quantities, and so reducing the *two* equations to *one* of *one* unknown quantity, is sufficient for the solution of any pair of equations of the above form which hold true

* When Equations are *bracketed* in this way it is meant that they hold true for the same values of the unknown quantities. They are sometimes called *Simultaneous* Equations.

for the same values of x and y. But there are other methods of *eliminating* one of the unknown quantities, which are less troublesome; thus,

(1) $\left.\begin{array}{r}\text{If } 2x+3y=20,\\ \text{and } 2x-3y=\ \ 8,\end{array}\right\}$ be the given equations,

adding the equal quantities together, we have

$$4x=28, \quad \therefore\ x=\frac{28}{4}=7.$$

$$\text{Subtracting,}\quad 6y=12, \quad \therefore\ y=\frac{12}{6}=2.$$

(2) $\left.\begin{array}{r}\text{Again, if } 2x+\ \ y=16,\\ \text{and } 3x+2y=25,\end{array}\right\}$ be the given equations,

multiplying the 1st equation by 2, we have

$$\left.\begin{array}{r}4x+2y=32,\\ \text{and from 2nd equation, } 3x+2y=25,\end{array}\right\}$$

subtracting, $x=7$. Also $y=16-2x$ (from 1st equation) $=16-14=2$.

(3) $\left.\begin{array}{r}\text{Or, again, if } 2x+3y=20,\\ \text{and } 3x+2y=25,\end{array}\right\}$ be the given equations,

$$\left.\begin{array}{l}\text{multiply 1st equation by 2, } 4x+6y=40,\\ \text{...... 2nd ...... by 3, } 9x+6y=75,\end{array}\right\}.$$

$$\text{Subtracting,}\quad 5x=35, \quad \therefore\ x=\frac{35}{5}=7.$$

Also, from 2nd equation,

$$2y=25-3x=25-21=4, \quad \therefore\ y=\frac{4}{2}=2.$$

The methods here employed may obviously be applied to all other like cases, where two distinct equations are given, either in the above form, or capable by previous rules of being reduced to that form:—the general object being so to frame *one* equation out of the two, that *one* of the unknown quantities shall be made to disappear. The most usual method is that employed in the *last* case; and the rule is—

Mark which of the unknown quantities has the least *coefficients*, (so as to make the easiest multipliers), and supposing it to be y, multiply the 1st equation by the coefficient of y in the 2nd, and the 2nd equation by the coefficient of y

in the 1st. Then the two resulting equations will be such that either by adding them together, or subtracting one from the other (it will easily be seen which), y will disappear altogether, and leave a simple equation of one unknown quantity, x; or *vice versâ*, if x be made to disappear.

Ex. 1. If $\left.\begin{array}{l} 2x + 16y = 48, \\ \text{and } 5x - 13y = 67, \end{array}\right\}$ find x and y.

Multiply 1st equation by 5, and the 2nd by 2, then

$$\left.\begin{array}{r} 10x + 80y = 240, \\ \text{and } 10x - 26y = 134, \end{array}\right\}$$

$$\text{Subtracting, } \quad 106y = 106,$$

$$\therefore y = 1.$$

$$\text{Also } 2x = 48 - 16y = 48 - 16 = 32,$$

$$\therefore x = 16.$$

That these are the correct values will thus appear:—

$$2x + 16y = 2 \times 16 + 16 \times 1 = 32 + 16 = 48,$$

$$\text{and } 5x - 13y = 5 \times 16 - 13 \times 1 = 80 - 13 = 67.$$

Ex. 2. If $\left.\begin{array}{l} 7x - 8y = 3, \\ \text{and } 13x + 5y = 85, \end{array}\right\}$ find x and y.

Here the coefficients of y are the smaller, $\therefore$ multiply the 1st equation by 5, the coefficient of y in 2nd equation, and the 2nd by 8, the coefficient of y in 1st equation, and we have

$$\left.\begin{array}{r} \text{from the 1st, } \quad 35x - 40y = 15, \\ \text{...... 2nd, } 104x + 40y = 680, \end{array}\right\}$$

$$\text{adding, } 139x = 695,$$

$$\therefore x = \frac{695}{139} = 5.$$

Also, from 1st equation, $8y = 7x - 3 = 35 - 3 = 32$,

$$\therefore y = \frac{32}{8} = 4.$$

That these are the correct values will easily appear *on trial:* for $7x = 35$, and $8y = 32$, $\therefore 7x - 8y = 35 - 32 = 3$. Also $13x = 65$, and $5y = 20$, $\therefore 13x + 5y = 65 + 20 = 85$.

[*Exercises* Q, 1...15, p. 86.]

There are some cases, however, in which the preceding Rule should not be strictly applied, especially if the *numerical* quantities in the equations are *large*. For example,

Ex. 1. If $\left.\begin{array}{r}16x+23y=94,\\ \text{and } 14x-12y=18,\end{array}\right\}$ find x and y.

Here 112 is the *Least Com. Mult.* of 16 and 14, and it contains the former 7 times and the latter 8 times; $\therefore$ multiplying the 1st equation by 7, and the 2nd by 8, we have

$$\left.\begin{array}{r}112x+161y=658,\\ 112x-\ \ 96y=144,\end{array}\right\}$$

$$\text{Subtracting,}\quad 257y=514,$$

$$\therefore\ y=\frac{514}{257}=2.$$

$$\text{Also }\ 14x=12y+18=24+18=42,$$

$$\therefore\ x=\frac{42}{14}=3.$$

Ex. 2. If $\left.\begin{array}{r}54x-121y=15,\\ \text{and } 36x-\ \ 77y=21,\end{array}\right\}$ find x and y.

Here 216 is the L. C. M. of 54 and 36, and it contains the former 4 times and the latter 6 times; $\therefore$ multiplying the 1st equation by 4, and the 2nd by 6, we have

$$\left.\begin{array}{r}216x-484y=\ \ 60,\\ 216x-462y=126,\end{array}\right\}$$

$$\text{Subtracting,}\quad 22y=66,$$

$$\therefore\ y=\frac{66}{22}=3.$$

$$\text{Also }\ 36x=21+77y=21+231=252,$$

$$\therefore\ x=\frac{252}{36}=7.$$

[*Exercises* Q, 16...20.]

EXERCISES. Q.

Find the values of x and y in the following equations:—

(1) $\left.\begin{array}{r}x+y=17,\\ 2x-y=19,\end{array}\right\}$

(2) $\left.\begin{array}{r}4x-7y=26,\\ 4x+5y=50,\end{array}\right\}$

(3) $\left.\begin{array}{r}5x+y=32,\\ 3x-2y=14,\end{array}\right\}$

(4) $\left.\begin{array}{r}3x-7y=2,\\ 11y-3x=2,\end{array}\right\}$

(5) $\left.\begin{array}{r} 3x + 4y = 11, \\ 15x - 2y = 11, \end{array}\right\}$

(6) $\left.\begin{array}{r} 13x - 6y = 31, \\ 11x - 3y = 47, \end{array}\right\}$

(7) $\left.\begin{array}{r} 7x - 6y = 10, \\ 6x - 7y = 3, \end{array}\right\}$

(8) $\left.\begin{array}{r} 35x + 2y = 76, \\ 12y - x = 34, \end{array}\right\}$

(9) $\left.\begin{array}{r} 5x + 2y = 16, \\ 9y + 2x = 31, \end{array}\right\}$

(10) $\left.\begin{array}{r} 11x - 7y = 72, \\ 7x - 11y = 0, \end{array}\right\}$

(11) $\left.\begin{array}{r} 36x - 45y = 0, \\ 2x + 5y = 1\frac{1}{2}, \end{array}\right\}$

(12) $\left.\begin{array}{r} 9x + 5y = 65, \\ 7x - 2\frac{1}{2}.y = 25, \end{array}\right\}$

(13) $\left.\begin{array}{r} 15x - y = 143, \\ 35y + x = 255, \end{array}\right\}$

(14) $\left.\begin{array}{r} 11x - 13y = 16, \\ 20x - 19y = 43, \end{array}\right\}$

(15) $\left.\begin{array}{r} 45x + 8y = 350, \\ 21y - 13x = 132, \end{array}\right\}$

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(16) $\left.\begin{array}{r} 101x - 24y = 63, \\ 103x - 28y = 29, \end{array}\right\}$

(17) $\left.\begin{array}{r} 64x + 90y = 237, \\ 63x - 218y = 80, \end{array}\right\}$

(18) $\left.\begin{array}{r} 3\frac{1}{2}.x - 4\frac{1}{2}.y = 12, \\ 7x + 9y = 60, \end{array}\right\}$

(19) $\left.\begin{array}{r} 2\frac{1}{4}.x + 3\frac{1}{3}.y = 48, \\ 4\frac{1}{2}.x + 10y = 126, \end{array}\right\}$

(20) $\left.\begin{array}{r} 4\frac{1}{4}.x - 3\frac{1}{2}.y = 6, \\ \frac{1}{2}x - \frac{1}{4}y = 2. \end{array}\right\}$

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56. When the equations are not given in the form of the foregoing examples, they must be reduced to that form by the rules before employed. Thus,

Ex. 1. If $\left.\begin{array}{r} 2(x+y) = 3(x-y) + 10, \\ \text{and } 2x - y = 4(2y - x) + 3, \end{array}\right\}$ find x and y.

From 1st equation, $2x + 2y = 3x - 3y + 10,$

or $5y - x = 10,$ .... (1)

From 2nd equation, $2x - y = 8y - 4x + 3,$

or $6x - 9y = 3,$

or $2x - 3y = 1$ .... (2)

The reduced equations, then, are $\left.\begin{array}{r} 5y - x = 10, \\ \text{and } 2x - 3y = 1, \end{array}\right\}$

Multiply (1) by 2, $\left.\begin{array}{r} 10y - 2x = 20, \\ 2x - 3y = 1, \end{array}\right\}$
but from (2)

adding, $7y = 21, \quad \therefore y = \frac{21}{7} = 3.$

Also, from (1), $x = 5y - 10 = 15 - 10 = 5.$

Ex. 2. If $\left.\begin{array}{l}\dfrac{2x-y}{3}+6=\dfrac{2y-x}{2}+\dfrac{9}{2}, \\ \text{and } \dfrac{3x+y}{5}+1=\dfrac{3y+x+13}{10},\end{array}\right\}$ find x and y.

To clear of fractions,

multiply 1st equation by 6, $4x-2y+36=6y-3x+27,$

transposing, $4x+3x-2y-6y=27-36,$

combining, $7x-8y=-9$.........(1)

Multiply 2nd equation by 10, $6x+2y+10=3y+x+13,$

transposing and combining, $5x-y=3$..........(2)

The two reduced equations, then, are

$$\left.\begin{array}{r}7x-8y=-9, \\ \text{and } 5x-y=3,\end{array}\right\}$$

Multiply the latter by 8, $\left.\begin{array}{l}40x-8y=24, \\ 7x-8y=-9,\end{array}\right\}$

and from (1),

subtracting, $33x=33,$

$\therefore\ x=1.$

Also, from (2), $y=5x-3=5-3=2.$

Ex. 3. If $\left.\begin{array}{l}\dfrac{3x-5y}{2}+3=\dfrac{2x+y}{5}, \\ \text{and } 8-\dfrac{x-2y}{4}=\dfrac{x}{2}+\dfrac{y}{3},\end{array}\right\}$ find x and y.

To clear of fractions,

Multiply 1st equation by 10, $15x-25y+30=4x+2y,$

transposing and combining, $11x-27y=-30$......(1)

Multiply 2nd equation by 12, $96-3x+6y=6x+4y,$

transposing and combining, $96=9x-2y$ ....(2)

Multiply (1) by 9, $\left.\begin{array}{l}99x-243y=-270, \\ 99x-22y=1056,\end{array}\right\}$

...... (2) by 11,

subtracting, $221y=1326,$

$$\therefore\ y=\frac{1326}{221}=6.$$

Also, from (2), $9x=96+2y=96+12=108,$

$$\therefore\ x=\frac{108}{9}=12.$$

EXERCISES. R.

Find the values of x and y in each of the following equations:—

(1) $\left.\begin{array}{l} 3(4x-5y)=2(x+y)+3, \\ 4(3x-2y)=5(x-y)+11, \end{array}\right\}$

(2) $\left.\begin{array}{l} 3x+\dfrac{y}{3}=36, \\ \dfrac{6y-2x}{4}=8, \end{array}\right\}$

(3) $\left.\begin{array}{l} \dfrac{3x-2y}{2}-3=\dfrac{2x-y}{4}, \\ \dfrac{5x-4y}{2}-3=\dfrac{4x-3y}{3}, \end{array}\right\}$

(4) $\left.\begin{array}{l} \dfrac{2x-3}{2}+y=7, \\ 5x-13y=33\frac{1}{2}, \end{array}\right\}$

(5) $\left.\begin{array}{l} \dfrac{x+3}{y}=\dfrac{1}{3}, \\ \dfrac{x}{y-1}=\dfrac{1}{5}, \end{array}\right\}$

(6) $\left.\begin{array}{l} \dfrac{x}{8}+\dfrac{y}{9}=42, \\ \dfrac{x}{9}+\dfrac{y}{8}=43, \end{array}\right\}$

(7) $\left.\begin{array}{l} \dfrac{x}{6}+\dfrac{y}{11}=26, \\ \dfrac{x}{2}-\dfrac{y}{7}=46, \end{array}\right\}$

(8) $\left.\begin{array}{l} \frac{1}{2}(x+y)=\frac{1}{3}(2x+4), \\ \frac{1}{3}(x-y)=\frac{1}{2}(x-24), \end{array}\right\}$

(9) $\left.\begin{array}{l} \frac{1}{7}(x+2)+\frac{1}{4}(y-x)=2x-8, \\ \frac{1}{3}(2y-3x)+\frac{1}{6}(8x+6y-4)=3x+4, \end{array}\right\}$

(10) $\left.\begin{array}{l} \frac{1}{3}(3x-7y)=\frac{1}{5}(2x+y+1) \\ 8-\frac{1}{6}(x-y)=6, \end{array}\right\}$

(11) $\left.\begin{array}{l} \dfrac{x-2}{5}-\dfrac{10-x}{3}=\dfrac{y-10}{4}, \\ \dfrac{2y+4}{3}-\dfrac{2x+y}{8}=\dfrac{x+13}{4}, \end{array}\right\}$

(12) $\left.\begin{array}{l} \dfrac{2x+y}{9}+\dfrac{7y+6x+11}{18}=9\frac{1}{2}-\dfrac{5x-17}{6}, \\ \frac{3}{7}(5x+3y+2)=\frac{1}{2}(9y+6). \end{array}\right\}$

PROBLEMS

DEPENDING UPON THE SOLUTION OF SIMPLE EQUATIONS OF TWO UNKNOWN QUANTITIES.

PROB. 1. The sum of two numbers is 26, and, if the half of the greater of them be added to the third part of the other, the sum of these parts is 11. What are the numbers?

Let x and y be the numbers required, then, by the question $x+y=26$.

Also $\frac{x}{2}=$ the half of one, and $\frac{y}{3}=$ the third part of the other, $\therefore$ by the question, $\frac{x}{2}+\frac{y}{3}=11$.

Thus, then, the problem is reduced to finding x and y from these two equations,

$$\left.\begin{aligned} x+y&=26, \\ \text{and } \frac{x}{2}+\frac{y}{3}&=11, \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{Multiply the 2nd by 6,} \quad 3x+2y&=66, \\ \text{.........} \quad \text{1st by 2,} \quad 2x+2y&-52, \end{aligned}\right\}$$

$$\text{subtracting, } x=14.$$

$$\text{Also } y=26-x=26-14=12.$$

$\therefore$ the numbers required are 14 and 12.

These values are correct, $\because$ $14+12=26$,

$$\text{and } \frac{14}{2}+\frac{12}{3}=7+4=11.$$

Note. It is not absolutely necessary to have *two* unknown quantities, x and y, here. The Problem may also be solved as follows:—

Let x be the greater of the two numbers,

then $26-x$ is the other, by the question,

$\therefore$ $\frac{x}{2}$ is the half of the greater.

and $\dfrac{26-x}{3}$ the third part of the other,

and $\dfrac{x}{2}+\dfrac{26-x}{3}=11$, by the question, which is a simple equation of *one* unknown quantity.

Multiply by 6, $3x+52-2x=66$,

$\therefore\ x=66-52=14$, one of the numbers;

and $26-x=26-14=12$, the other.

Prob. 2. I have as many shillings, and pennies, as together make £1. 5*s*. 0*d*., and if the pennies were shillings and the shillings pennies, then I should have only 14*s*. How many have I of each?

Let x be the number of shillings,

y .... pennies,

then x shillings $=12x$ pence,

£1. 5*s*. $=300$ pence,

$\therefore\ 12x+y=300$, by the question, (1).

Again, y shillings $=12y$ pence,

14*s*. $=$ 168 pence,

$\therefore\ 12y+x=$ 168, by the question.

From (1) $144x+12y=3600$,

$\therefore$ subtracting, $143x=3432$,

$\therefore\ x=\dfrac{3432}{143}=24$, the number of shillings,

and $y=300-12x=\left.\begin{array}{r}300\\-288\end{array}\right\}=12$, the number of pennies.

Prob. 3. *Seven* years ago a father was 4 times as old as his son, but in *seven* years more he will be only *twice* as old. What is the age of each?

Let x be the son's age,

y the father's age,

then $x-7=$ son's age 7 years ago,

$y-7=$ father's.................

$x+7=$ son's age in 7 years more,

$y+7=$ father's.......................

$\therefore$ by the question,

$$\left.\begin{aligned} y-7&=4(x-7), \\ \text{and } y+7&=2(x+7), \end{aligned}\right\} \text{ from which equations it remains to find } x \text{ and } y.$$

$$\left.\begin{aligned} \text{or } y-7&=4x-28, \\ y+7&=2x+14, \end{aligned}\right\}$$

subtracting, $-14=2x-42$,

$$\therefore\ 2x=42-14=28,$$

$$\therefore\ x=\frac{28}{2}=14, \text{ the son's age,}$$

and $y=7+4(x-7)=7+28=35$, the father's age.

Prob. 4. I have in my purse a sum of money consisting of sovereigns and half-crowns: If I had twice as many sovereigns and half as many half-crowns, I should have £20. 10*s.*; but if I had half as many sovereigns, and twice as many half-crowns, I should have only £7. What is the number of each coin?

Let x be the number of sovereigns,

and y .... half-crowns.

Then $\because$ $2x$ sovereigns $=20\times 2x$ shillings $=40x$ shillings,

and $\frac{y}{2}$ half-crowns $=2\frac{1}{2}\times\frac{y}{2}$ .... $=\frac{5y}{4}$ ....

and £20. 10*s.* $=410$ shillings,

$\therefore$ by the question, $40x+\frac{5y}{4}=410$,

or dividing by 5, $8x+\frac{y}{4}=82$,

or $32x+y=328$ ....(1).

Again, $\frac{x}{2}$ sovereigns $=20\times\frac{x}{2}$, or $10x$ shillings,

and $2y$ half-crowns $=2\frac{1}{2}\times 2y$, or $5y$ shillings,

$\therefore$ by the question, $10x+5y=140$,

$$\left.\begin{aligned} \text{or } 2x+y&=\ 28, \\ \text{but from (1) } 32x+y&=328, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$\therefore$ subtracting, $30x=300$;

$$\therefore\ x=\frac{300}{30}=10, \text{the number of sovereigns.}$$

Also from (2) $y = 28 - 2x = 28 - 20 = 8$, the number of half-crowns.

PROB. 5. An orange-woman bought oranges, and afterwards forgot the price; but she recollected, that she paid for them in shillings and halfpence—that the number of each coin was the same—and that she had as many dozens of oranges as the number of shillings and halfpence taken together. What was the price per dozen?

Let x be the price per doz. *in pence*,
y the number of shillings paid, and also the number of halfpence,
then $2y =$ number of dozens of oranges, by the question,
and $2y \times x$ or $2xy =$ cost of all the oranges, *in pence*,
but the cost of the whole is y shillings + y halfpence,

$$\text{or } 12y + \frac{y}{2} \textit{ pence},$$

$$\therefore\ 12xy = 12y + \frac{y}{2},$$

$2x = 12 + \frac{1}{2}$, dividing by y, which is in every term,
$\therefore\ x = 6 + \frac{1}{4}$, or $6\frac{1}{4}d.$, the price per doz. required.

In this solution *two* unknown quantities have been employed, but one only being *required*, and the Problem not furnishing a *second* equation, the other *disappeared* by division. It may serve to shew the *convenience* of sometimes using two unknown quantities to obtain the value of *one* only.

Another method of solution is as follows, assuming x to represent, not the number *required*, but the number of each coin paid for the oranges; then

the *whole cost* of the oranges $= x$ shillings $+ x$ halfpence,

$$= 12x + \frac{x}{2} \textit{ pence},$$

and the number of dozens ... $= x + x$, or $2x$, by the question,

$$\therefore \text{ price per dozen } \ldots = \frac{\text{whole cost}}{\text{number of dozens}},$$

$$= \frac{12x + \frac{x}{2}}{2x} = 6 + \frac{1}{4},$$

$$= 6\tfrac{1}{4}d$$

PROB. 6. A certain fraction becomes 1, if 1 be added to its numerator; but if 2 be added to its denominator, it becomes $\frac{1}{2}$. Find the fraction.

Let $\frac{x}{y}$ be the fraction required; then, adding 1 to the numerator, the fraction becomes $\frac{x+1}{y}$,

$$\therefore \text{ by the question. } \frac{x+1}{y} = 1;$$

or, multiplying by y, $x + 1 = y$, ... (1).

Also, by the question, $\frac{x}{y+2} = \frac{1}{2}$, $\therefore\ 2x = y + 2$, ...(2).

But from (1), $y = x + 1$, $\therefore\ 2x = x + 1 + 2$, $\therefore\ x = 3$.

And $y = x + 1$, $\therefore\ y = 4$, $\therefore\ \frac{x}{y} = \frac{3}{4}$, the fraction required.

PROB. 7. There is a certain number composed of two figures or digits, which is equal to four times the sum of its digits; and if the digits exchange places the number thus formed is less by 12 than twice the former number. What is the number?

Let x be the digit in the *tens'* place,
y .... *units'* ...

then $10x + y$ is the number, (just as $23 = 10 \times 2 + 3$,) $\therefore$ by the question,

$$\begin{aligned} 10x + y &= 4(x + y), \\ &= 4x + 4y, \\ 10x - 4x &= 4y - y, \\ 6x &= 3y, \\ 2x &= y \text{ ....(1).} \end{aligned}$$

Again, if the digits be reversed, $10y + x$ will be the number, $\therefore$ by the question,

$$\begin{aligned} 10y + x &= 2(10x + y) - 12, \\ &= 20x + 2y - 12, \\ 19x - 8y &= 12, \\ 19x + 16x &= 12, \quad \because y = 2x, \text{ from (1)}, \\ 3x &= 12, \\ \therefore\ x &= 4; \text{ and } y = 2x = 8. \end{aligned}$$

$\therefore$ the number required is 48.

PROB. 8. Iron, worth £10. in its raw state, is manufactured half into knife-blades and half into razors, and is then worth £444. But if *one-third* of it had been made into razors and the rest into knife-blades, the produce would have been worth £30. more than in the former case. How much is the value of the original material increased by these respective manufactures?

Let £1. in raw iron become x£. in knife-blades,

and .... y£....razors;

then, ∵ £5. in raw iron is made into knife-blades, and £5. also into razors, by the question,

$$5x + 5y = 444 \ldots\ldots\ldots\ldots(1)$$

Again, on second supposition, $\frac{1}{3}$ of 10£. in raw iron, that is, $\frac{10}{3}$£. is made into razors; and $\frac{2}{3}$ of 10£., that is, $\frac{20}{3}$£. is made into knife-blades; ∴ by the question,

$$\frac{20}{3}x + \frac{10}{3}y = 444 + 30,$$

$$\text{or } 20x + 10y = 1422 \ldots\ldots\ldots\ldots(2)$$

Now from (1) $10x + 10y = 888$,

∴ subtracting from (2), $10 = 534$,

$$\therefore x = \frac{534}{10} = 53\tfrac{2}{5} = £53.\ 8s.$$

Also from (1), $5y = 444 - 5x = 444 - 267 = 177$,

$$\therefore y = \frac{177}{5} = 35\tfrac{2}{5} = £35.\ 8s.$$

Hence, every pound's worth of raw iron is increased in value to £53. 8*s.* if made into knife-blades; and to £35. 8*s.* if made into razors.

EXERCISES. S.

1. Says Charles to William, If you give me 10 of your marbles, I shall then have just *twice* as many as you: but says William to Charles, If you give me 10 of yours, I shall then have *three times* as many as you. How many had each?

2. A man, who has two purses containing money, receives £10. to add to them, and finds that if he puts £5. into each, one will then contain exactly twice as much as

the other, but if he puts the whole £10. into that which already contains the most, its contents will be just *three times* the value of the other. How much was there in each purse to begin with?

3. A party consists of men and women, and there are 6 men to every 5 women; but if there had been 2 men less and 2 women more, the number of each would have been the same. How many are there?

4. A clergyman, who had a dole of £5. 10*s*. to distribute amongst a certain number of old men and widows, found that, if he gave them 3*s*. each, he would be 1*s*. out of pocket; but, if he gave each of the men 2*s*. 2*d*. and each of the widows 3*s*. 6*d*., he would have 6*d*. to spare. How many were there of each?

5. There is a certain fraction which becomes equal to $\frac{1}{2}$, when both numerator and denominator are diminished by 1; but, if 2 be taken from the numerator and added to the denominator, it becomes equal to $\frac{1}{3}$. What is the fraction?

6. What is the fraction in which twice the sum of the numerator and denominator is equal to three times their difference?

7. Find two numbers such that one shall be as much above 10, as the other is below it, and one-tenth of their sum equal to one-fourth of their difference.

8. Find two numbers such that the half of one added to a third of the other is 12, but a third of the former added to half the other is 13.

9. A person has two casks with a certain quantity of wine in each. He draws out of the first into the second as much as there was in the second to begin with: then he draws out of the second into the first as much as was left in the first: and then again out of the first into the second as much as was left in the second. There are then exactly 8 gallons in each cask. How much was there in each at first?

10. In the course of last century the change took place, called '*the change of Style*', which consisted in beginning the year with Jan. 1, instead of March 25, as heretofore, and for that year only, calling the day after Sep. 2, the 14th, instead of the 3rd. Now the year of our Lord in which this happened, possesses the following properties:—The first digit being 1 for thousands, the second is the sum of the third and fourth, the third is the *third* part of the sum of all four, and the fourth is the *fourth* part of the sum of the first two. Determine the year.

INVOLUTION AND EVOLUTION.

57. **Def.** A quantity multiplied by itself once, or successively more than once, is said to be *involved*, or *raised to a certain power;* and the *power* to which it is raised is marked by the number of times the quantity occurs as a *factor* in the multiplication.

Thus $a \times a$, or a^2, expresses that a is raised to the 2nd *power*, because a occurs *twice* as a factor; and so on. See Art. 9.

Involution, therefore, differs not in reality from *Multiplication*, and requires no rules different from those already given.

It may, however, be worth while to *observe* here the *particular results* in certain cases, when Multiplicand and Multiplier, as in Involution, are both alike. Thus,

1st. Any *simple* quantity, of one letter, as a, is raised to the 2nd *power*, or *squared*, by *doubling* its index. For example,

a, or a^1, squared is a^2,

a^2 .... a^4, $\because a^2 \times a^2 = a^{2+2} = a^4$, (Art. 24.)

a^3 .... a^6, $\because a^3 \times a^3 = a^{3+3} = a^6$, ...

and so on.

2nd. Any *product*, or quantity of two factors, as ab, is raised to the 2nd power, or *squared*, by squaring each factor separately, and taking the product of those results. For example,

ab squared is a^2b^2, $\because ab \times ab = abab = aabb$ (Art. 5) $= a^2b^2$;

a^2b .... a^4b^2, $\because a^2b \times a^2b = a^2ba^2b = a^2a^2bb = a^4b^2$;

ab^2 .... a^2b^4, $\because ab^2 \times ab^2 = ab^2ab^2 = aab^2b^2 = a^2b^4$;

and so on.

Similarly, $3xy$ squared $= 3xy \times 3xy = 3 \times 3xxyy = 9x^2y^2$;

$2abc$ squared $= 4a^2b^2c^2$; squaring each factor *separately*, whatever be the number of them.

3rd. Any *fraction*, as $\frac{a}{b}$, is *squared* by squaring the numerator and denominator separately. For example,

$\frac{a}{b}$ squared is $\frac{a^2}{b^2}$, $\because$ $\frac{a}{b} \times \frac{a}{b} = \frac{aa}{bb}$ (Art. 40) $= \frac{a^2}{b^2}$;

$\frac{ab}{cd}$ .... $\frac{a^2b^2}{c^2d^2}$, $\because$ $\frac{ab}{cd} \times \frac{ab}{cd} = \frac{ab \times ab}{cd \times cd}$ (Art. 40) $= \frac{a^2b^2}{c^2d^2}$;

$\frac{2x}{3y}$ .... $\frac{4x^2}{9y^2}$; and so on, whatever the fraction be.

[*Exercises* T, 1...11, p. 99.]

4th. Any quantity of *two terms, both positive,* as $a + b$, is squared by squaring each term separately, and *adding* to the *sum* of these twice the *product* of the two terms. For

$\overline{a+b}$ squared is $a^2 + b^2 + 2ab$, See Art. 23, Ex. 4,

that is, the square of a + the square of b + twice the *product* of a and b.

5th. Any quantity of *two terms*, one of which is *negative*, as $a - b$, is squared by squaring each term separately, and *subtracting* from the sum of these twice the product of the two terms. For

$\overline{a-b}$ squared is $a^2 + b^2 - 2ab$, See Art. 23, Ex. 5.

This case comes under the same rule as the preceding one, if the quantities are taken *along with their proper signs.*

Ex. 1. $(1 + x)^2 = 1^2 + x^2 + 2 \times 1 \times x = 1 + x^2 + 2x$.

Ex. 2. $(1 - x)^2 = 1^2 + x^2 - 2 \times 1 \times x = 1 + x^2 - 2x$.

Ex. 3. $(2 + x)^2 = 2^2 + x^2 + 2 \times 2 \times x = 4 + x^2 + 4x$.

Ex. 4. $(2x - y)^2 = (2x)^2 + y^2 - 2 \times 2x \times y = 4x^2 + y^2 - 4xy$.

Ex. 5. $(2a + 3b)^2 = (2a)^2 + (3b)^2 + 2 \times 2a \times 3b = 4a^2 + 9b^2 + 12ab$.

Ex. 6. $(ab - 1)^2 = (ab)^2 + 1^2 - 2 \times ab \times 1 = a^2b^2 + 1 - 2ab$.

[*Exercises* T, 12...24, p. 99.]

58. The last two Rules may be used with effect sometimes in *Mental Arithmetic*, as it is called, which means Arithmetic worked in the mind and memory, *without writing*. Thus suppose the square of 25 be required; since $25 = 20 + 5$,

$\therefore$ square of $25 =$ square of $\overline{20 + 5}$,

$=$ square of 20 + square of 5 + twice product of 20 and 5,

$= 400 + 25 + 200$,

$= 625$,

all which may be readily done in the mind without writing.

Again, the square of $15 =$ the square of $\overline{10 + 5}$,

$$= 10^2 + 5^2 + 2 \times 5 \times 10,$$
$$= 100 + 25 + 100,$$
$$= 225.$$

The use of this method, however, will be best seen in larger numbers: thus, required the square of 499; since $499 = 500 - 1$,

$\therefore$ square of $499 =$ square of $\overline{500 - 1}$,

$$= \text{square of } 500 + \text{square of } 1 - 2 \times 500 \times 1,$$
$$= 250{,}000 + 1 - 1{,}000,$$
$$= 249{,}000 + 1,$$
$$= 249{,}001,$$

all which may be readily done *in the mind* without actual writing.

59. It is to be observed that a quantity of *one term* squared is still of one term; and a quantity of *two terms* squared produces a quantity of *three terms*. Hence *no quantity of two terms* can have been produced by squaring, that is, *can be a complete square.*

It should also not be forgotten, that, although the square of $a \times b$ is $a^2 \times b^2$, the square of $a + b$ is not $a^2 + b^2$, but $a^2 + b^2 + 2ab$, a and b representing any quantities whatever.

EXERCISES. T.

Square each of the following quantities:—

(1) $5ax$.

(2) $5axy$.

(3) $-7ab$.

(4) a^2bc.

(5) $-7a^2bc^3$.

(6) $\dfrac{ab}{c}$.

(7) $\dfrac{3ax}{2by}$.

(8) $\dfrac{a^2b}{2c}$.

(9) $\dfrac{4a^2b}{7x^3y^3}$.

(10) $\dfrac{-3xy^2}{2x^3}$.

(11) $\dfrac{4}{5a^2bc^3}$.

(12) $a + 1$.

(13) $ab + 1$.

(14) $x + 3$.

(15) $2 - y$.

(16) $2m - n$.

(17) $2x - 3y$.

(18) $x - \dfrac{p}{2}$.

(19) $x + \dfrac{3}{2}$.

(20) $mx + n$.

(21) $2mx - n$.

(22) $abx + c$.

(23) $3xy - a$.

(24) $\frac{1}{2}ab + c$.

60. EVOLUTION is precisely the reverse operation to *Involution.* We have to *evolve*, or *extract*, the quantity called the *root*, by the *involution* of which the proposed quantity is produced. Thus, to *evolve*, or *extract*, the *square root* of 25, is to find the number which being *squared* produces 25; that is, 5. Hence the *square root* of a^2 is a, because a is the quantity which being *squared* produces a^2; and so on.

61. To extract the square root of a simple quantity, of one letter, as a, we must *halve* its index. Thus,

the square root of a^2 is a^1 or a, $\because a \times a = a^2$,
the square root of a^4 is a^2, $\because a^2 \times a^2 = a^4$;

and so on.

62. To extract the square root of any *product*, of two factors, we must extract the square root of each factor separately, and take the product of these results. Thus writing $\sqrt{\ }$ for 'the square root of',

$\sqrt{ab} = \sqrt{a} \,.\, \sqrt{b}$, $\because \sqrt{a} \,.\, \sqrt{b} \times \sqrt{a} \,.\, \sqrt{b} = \sqrt{a} \,.\, \sqrt{a} \,.\, \sqrt{b} \,.\, \sqrt{b} = ab$,
$\sqrt{a^2b} = \sqrt{a^2} \,.\, \sqrt{b}$, $\because \sqrt{a^2} \,.\, \sqrt{b} \times \sqrt{a^2} \,.\, \sqrt{b} = \sqrt{a^2} \,.\, \sqrt{a^2} \,.\, \sqrt{b} \,.\, \sqrt{b} = a^2b$;

and so on; from which it appears that $\sqrt{a} \,.\, \sqrt{b}$ *squared* produces ab, and $\therefore \sqrt{a} \,.\, \sqrt{b}$ is the *square root* of ab; and similarly for any other *product* of *two factors*.

By the same method of reasoning it may be shewn that the square root of a *product of three or more factors* is found by taking the square root of *each factor separately*. Thus $\sqrt{abc} = \sqrt{a} \,.\, \sqrt{b} \,.\, \sqrt{c}$; and so on.

[*Exercises* U, 1...3, p. 103.]

63. To extract the square root of a *fraction* we must take the square root of the numerator and denominator *separately*. Thus

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}, \quad \because \frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a} \,.\, \sqrt{a}}{\sqrt{b} \,.\, \sqrt{b}} \text{ (Art. 40)} = \frac{a}{b},$$

which shews that $\frac{\sqrt{a}}{\sqrt{b}}$ is the quantity which being *squared* produces $\frac{a}{b}$, and therefore it is the *square root* of $\frac{a}{b}$.

Exs. $\sqrt{\dfrac{100}{49}} = \dfrac{\sqrt{100}}{\sqrt{49}} = \dfrac{10}{7}$. $\sqrt{\dfrac{9a^2}{4x^2}} = \dfrac{\sqrt{9a^2}}{\sqrt{4x^2}} = \dfrac{3a}{2x}$.

[*Exercises* U, 4...6, p. 103.]

64. To extract the square root of a "*complete square*"* *of three terms*, arrange the terms according to the *powers* of some one letter, (as in *Division*), and take the sum or difference of the square roots of the extreme terms, taken separately, accordingly as the sign of the *middle term* is + or −. Thus, $a^2+2ax+x^2$ is a complete square arranged according to powers of a, and its square root is $\sqrt{a^2}+\sqrt{x^2}$, or $a+x$, $\because a+x$ squared produces $a^2+2ax+x^2$. The square root of $a^2-2ax+x^2$ is $a-x$, for the same reason.

Ex. 1. $\sqrt{a^2+1+2a}=\sqrt{a^2+2a+1}=\sqrt{a^2}+\sqrt{1}=a+1.$

Ex. 2. $\sqrt{x^2+9-6x}=\sqrt{x^2-6x+9}=\sqrt{x^2}-\sqrt{9}=x-3.$

Ex. 3. $\sqrt{4+y^2-4y}=\sqrt{y^2-4y+4}=\sqrt{y^2}-\sqrt{4}=y-2.$

Ex. 4. $\sqrt{x^2-px+\frac{p^2}{4}}=\sqrt{x^2}-\sqrt{\frac{p^2}{4}}=x-\frac{p}{2}.$

Ex. 5. $\sqrt{x^2+3x+\frac{9}{4}}=\sqrt{x^2}+\sqrt{\frac{9}{4}}=x+\frac{3}{2}.$

Ex. 6. $\sqrt{m^2x^2+2mnx+n^2}=\sqrt{m^2x^2}+\sqrt{n^2}=mx+n.$

Ex. 7. $\sqrt{9x^2y^2-6axy+a^2}=\sqrt{9x^2y^2}-\sqrt{a^2}=3xy-a.$

Ex. 8. $\sqrt{\frac{1}{4}a^2b^2+abc+c^2}=\sqrt{\frac{1}{4}a^2b^2}+\sqrt{c^2}=\frac{1}{2}ab+c.$

[*Exercises* U, 7...12, p. 103.]

65. Since either $+a$, or $-a$, multiplied by itself, produces a^2, therefore strictly speaking, the square root of a quantity has always a *double sign*, which is written thus $\pm$, and is read '*plus or minus*'. Thus $\sqrt{a^2}$ is $\pm a$; $\sqrt{a^2b^2}$ is $\pm ab$, $\sqrt{a^2+2ax+x^2}$ is $\pm(a+x)$; and so on.

To shew that $-(a+x)$ is the square root of $a^2+2ax+x^2$, as much as $a+x$ is, let us multiply $-(a+x)$ by itself; 1st. Removing the brackets, by Art. 44, $-(a+x)=-a-x$, and this multiplied by itself as under

$$\begin{array}{r} -a-x \\ -a-x \\ \hline a^2+ax \\ +ax+x^2 \\ \hline a^2+2ax+x^2 \end{array}$$

* By a "complete square" is meant a quantity which has been produced, or may be produced, by *squaring* some other quantity, and which therefore has an *exact* square root. Thus 25 is a 'complete square', and 26 is not.

produces $a^2+2ax+x^2$, $\therefore -a-x$, or $-(a+x)$ is the *square root* of $a^2+2ax+x^2$.

66. By observing the relation which exists between the parts of complete squares of three terms arranged according to the powers of one letter, as $x^2+2ax+a^2$, $x^2-px+\frac{p^2}{4}$, x^2+6x+9, &c., we see that the square of the middle term is always equal to 4 times the product of the extreme terms, and that no three terms can form a *complete square* which does not fulfil this condition.

Exs. $x^2-7x+16$ is not a *complete square*, although x^2 and 16 are both squares, because $(7x)^2$ or $49x^2$ is not equal to $4\times16x^2$. But $x^2-8x+16$ is a *complete square*, viz. the square of $x-4$, since $(8x)^2$ or $64x^2=4\times16x^2$.

Hence, if in any proposed case we are allowed to add to *two terms* another which shall make the *three*, when arranged according to the powers of one letter, a *complete square*, the added term must be such that the square of the middle term is equal to 4 times the product of the extremes.

For example, if x^2+px is to be made a *complete square* by adding another term, suppose the unknown term to be y, then, by the supposition, x^2+px+y is a *complete square*, and by the rule which applies to all *complete squares* of three terms, $(px)^2$ or $p^2x^2=4yx^2$, $\therefore y=\frac{p^2}{4}$, and

$$\therefore x^2+px+\frac{p^2}{4} \text{ is the } \textit{complete square.}$$

Similarly, if to x^2-px there be added $\frac{p^2}{4}$, the resulting quantity $x^2-px+\frac{p^2}{4}$ is a *complete square*, viz. the square of $x-\frac{p}{2}$.

Exs. To x^2+6x add $\left(\frac{6}{2}\right)^2$, or 3^2, and the root is $x+3$.

To x^2-8x ... $\left(\frac{8}{2}\right)^2$, or 4^2, ... $x-4$.

x^2-5x ... $\left(\frac{5}{2}\right)^2$, ... $x-\frac{5}{2}$.

To $x^2 + \frac{2}{3}x$ add $\left(\frac{1}{3}\right)^2$, and the root is $x + \frac{1}{3}$.

To $x^2 - \frac{3}{2}x$ ... $\left(\frac{3}{4}\right)^2$, ... $x - \frac{3}{4}$.

[*Exercises* U, 13...24.]

EXERCISES. U.

Extract the square root of each of the following quantities:—

(1) $4a^2b^2$.

(2) $9x^2y^4$.

(3) $100a^2b^4c^6$.

(4) $\frac{9a^2x^2}{4b^2}$.

(5) $\frac{4a^2b^2}{9x^2y^4}$.

(6) $\frac{1}{4} \cdot \frac{m^2x^4}{n^2y^2}$.

(7) $1 + x^2 - 2x$.

(8) $4x^2 + 4x + 1$.

(9) $4a^2 + b^2 - 4ab$.

(10) $9x^2 + 6x + 1$.

(11) $x^2 + x + \frac{1}{4}$.

(12) $x^2 + \frac{1}{x^2} - 2$.

Complete the squares in each of the following cases:—

(13) $x^2 - 12x$.

(14) $x^2 - 14x$.

(15) $x^2 + 11x$.

(16) $x^2 + 2x$.

(17) $x^2 - x$.

(18) $x^2 + \frac{4x}{5}$.

(19) $x^2 - \frac{2x}{7}$.

(20) $x^2 + \frac{1}{2}x$.

(21) $x^2 - \frac{1}{3}x$.

(22) $x^2 - \frac{5}{6}x$.

(23) $x^2 - \frac{3x}{4}$.

(24) $x^2 - \frac{7x}{10}$.

QUADRATIC EQUATIONS.

67. Def. There are two sorts of *Quadratic* Equations. 1st. Those which, either at first, or after reduction by the rules of Arts. 46...49, contain no other power of the unknown quantity but the 2nd, as x^2,—these are called *Pure Quadratics*. 2nd. Those which contain no other powers of the unknown quantity but the *first and second*, as x *and* x^2,—these are sometimes called *Adfected Quadratics*.

68. *Pure Quadratics* are solved precisely as *Simple* Equations, considering x^2 as the quantity sought in the first instance. Having found the value of x^2, it then remains only to extract the *square root* of the equal quantities, and x is found. Or the unknown quantity may be so involved as to present an equation (either at first, or by reduction,) of the form $(x-a)^2 = b$; then extracting the square root, we have $x - a = \pm\sqrt{b}$, and $\therefore\ x = a \pm \sqrt{b}$.

Ex. 1. If $3x^2 - 2 = 2x^2 + 2$, find x.

Transposing, $3x^2 - 2x^2 = 2 + 2$,

combining, $x^2 = 4$,

$$\therefore\ x = \sqrt{4} = \pm 2. \quad \text{(Art. 65).}$$

Ex. 2. If $\dfrac{x^2}{3} - \dfrac{x^2}{4} - \dfrac{x^2}{16} = \dfrac{1}{3}$, find x.

To clear off fractions, multiply by 48, the Least Common Multiple of the Denominators,

$$16x^2 - 12x^2 - 3x^2 = 16,$$

combining, $x^2 = 16$,

$$\therefore\ x = \sqrt{16} = \pm 4.$$

Ex. 3. If $7(2x^2 - 6) + 5(3 - x^2) = 198$, find x.

Here $7(2x^2 - 6) = 14x^2 - 42$, and $5(3 - x^2) = 15 - 5x^2$,

$\therefore$ erasing brackets (Art. 44), $14x^2 - 42 + 15 - 5x^2 = 198$,

transposing, $14x^2 - 5x^2 = 198 + 42 - 15$,

combining, $9x^2 = 225$,

dividing, $x^2 = \dfrac{225}{9} = 25$,

$$\therefore\ x = \sqrt{25} = \pm 5.$$

Ex. 4. If $\dfrac{4}{3+x}+\dfrac{4}{3-x}=3$, find x.

Multiply by $3+x$, $\quad 4+\dfrac{12+4x}{3-x}=9+3x$,

transposing, $\quad \dfrac{12+4x}{3-x}=5+3x$,

multiply by $3-x$, $\quad 12+4x=15+9x-5x-3x^2$,

transposing, $\quad 3x^2+4x+5x-9x=15-12$,

combining, $\quad 3x^2=3$,

dividing, $\quad x^2=1$,

$\therefore\ x=\pm 1$.

Ex. 5. If $(4x-5)^2=4x^2$, find x.

Extracting root, $\quad 4x-5=\pm 2x$,

$4x\mp 2x=5$, ($\mp$ is read '*minus or plus*'),

$\therefore\ 2x=5, \quad$ or $\ 6x=5$,

$\therefore\ x=\dfrac{5}{2}=2\frac{1}{2}, \quad$ or $\ x=\dfrac{5}{6}$.

EXERCISES. V.

Find the values of x in each of the following equations:—

(1) $3x^2-5=\dfrac{8x^2}{3}+7$.

(2) $(x+1)^2=2x+17$.

(3) $(x+2)^2=4x+5$.

(4) $(2x-5)^2=x^2-20x+73$.

(5) $x^2-\dfrac{3x^2-2}{5}=3-\dfrac{2x^2-5}{3}$.

(6) $\dfrac{2x^2+10}{15}=7-\dfrac{50+x^2}{25}$.

(7) $\dfrac{x^2}{5}-\dfrac{x^2}{15}+\dfrac{x^2}{25}=4\frac{1}{3}$.

(8) $13\frac{3}{4}-\dfrac{x^2}{2}=2x^2-8\frac{3}{4}$.

(9) $\dfrac{3}{1+x}+\dfrac{3}{1-x}=8$.

(10) $\dfrac{1}{x^2}-\dfrac{2}{3x^2+1}=\dfrac{5}{4(3x^2+1)}$.

(11) $\dfrac{14x^2+16}{21}-\dfrac{2x^2+8}{8x^2-11}=\dfrac{2x^2}{3}$.

(12) $\left(x-\dfrac{3}{4}\right)^2=\dfrac{1}{4}$.

69. ADFECTED QUADRATICS are solved by the following Rule:—

1st. Employ the methods given in Arts. 46...49 for *clearing*, *transposing*, *combining*, &c. until the equation is reduced to three terms, in the form $ax^2 + bx = c$, having collected all the terms containing x^2 into one, as ax^2, and all those containing x into one, as bx, to form one side of the equation, and placing the known quantities, as c, on the other.

2nd. Divide the whole equation by the coefficient of x^2, bringing it into the form $x^2 + \frac{b}{a}x = \frac{c}{a}$, replacing $\frac{b}{a}$, and $\frac{c}{a}$, by whole numbers, if they admit of it.

3rd. Add to each side the square of half the *coefficient* of x, which will make the left side a *complete square*. (Art. 66).

4th. Extract the square root of each side, and the result will be a *simple* equation, from which x is readily found.

Ex. 1. If $3x^2 - 12x + 32 = x^2 + 12x - 32$, find x.

Transposing, $3x^2 - x^2 - 12x - 12x = -32 - 32$,

combining, $2x^2 - 24x = -64$,

dividing by 2, $x^2 - 12x = -32$,

adding $\left(\frac{12}{2}\right)^2$, or 6^2, $x^2 - 12x + 6^2 = 36 - 32 = 4$,

extracting root, $x - 6 = \pm 2$,

$\therefore x = 6 \pm 2 = 8$, or 4.

Upon substituting these values for x in the original equation both of them are found to satisfy it.

Ex. 2. If $5(x^2 - 5) - 2x(x - 1) = 60$, find x.

Since $5(x^2 - 5) = 5x^2 - 25$, and $2x(x - 1) = 2x^2 - 2x$,

$\therefore 5x^2 - 25 - 2x^2 + 2x = 60$, (Art. 44,)

transposing, $5x^2 - 2x^2 + 2x = 60 + 25$,

combining, $3x^2 + 2x = 85$,

dividing by 3, $x^2 + \frac{2}{3}x = \frac{85}{3}$,

$$\text{adding } \left(\frac{1}{3}\right)^2, \quad x^2+\frac{2}{3}x+\left(\frac{1}{3}\right)^2=\frac{85}{3}+\frac{1}{9}=\frac{255+1}{9}=\frac{256}{9},$$

$$\text{extracting root,} \qquad x+\frac{1}{3}=\sqrt{\frac{256}{9}}=\pm\frac{16}{3},$$

$$\therefore\ x=\pm\frac{16}{3}-\frac{1}{3}=\frac{15}{3}, \text{ or } -\frac{17}{3},$$

$$=5, \text{ or } -5\tfrac{2}{3}.$$

Ex. 3. If $x^2+px=q$, find x.

$$\text{Adding } \left(\frac{p}{2}\right)^2, \quad x^2+px+\left(\frac{p}{2}\right)^2=\left(\frac{p}{2}\right)^2+q,$$

$$=\frac{p^2}{4}+q,$$

$$\text{extracting root,} \quad x+\frac{p}{2}=\pm\sqrt{\frac{p^2}{4}+q},$$

$\because$ the root of $\frac{p^2}{4}+q$ can only be thus *expressed*, (Art. 59).

$$\therefore\ x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}.$$

This being true whatever numbers p and q stand for, it is called the *general* solution of a quadratic equation, as it includes every *particular* equation of that form.

Thus, if $x^2+4x=12$, here $p=4$, and $q=12$,

$\therefore x=-\frac{4}{2}\pm\sqrt{\frac{16}{4}+12}$ {substituting for p and q in the above result their values in this particular case,

$=-2\pm\sqrt{16}=-2\pm4=2$, or -6.

And in *every* such equation it will be easy to write down at once the values of x by remembering the *general solution*, leaving only a little *arithmetical* working to simplify them.

Ex. 4. If $\frac{x+1}{x-1}-\frac{x-1}{x+1}=1$, find x.

Clearing fractions, $x^2+2x+1-(x^2-2x+1)=x^2-1$,

$$x^2+2x+1-x^2+2x-1=x^2-1,$$

combining, $\qquad x^2-4x=1,$

adding $\left(\frac{4}{2}\right)^2$, or 4, $\quad x^2-4x+4=5,$

extracting root, $\qquad x-2=\pm\sqrt{5},$

$$\therefore\ x=2\pm\sqrt{5}.$$

Ex. 5. If $\frac{1}{x} + \frac{1}{x+1} = \frac{1}{x+2}$, find x.

$$\frac{x+1+x}{x^2+x} = \frac{1}{x+2},$$
$$(2x+1)(x+2) = x^2 + x,$$
$$2x^2 + 5x + 2 = x^2 + x,$$
$$x^2 + 4x = -2,$$
$$x^2 + 4x + 4 = 4 - 2 = 2,$$
$$x + 2 = \pm\sqrt{2},$$
$$\therefore x = -2 \pm \sqrt{2}.$$

70. There is another method of "*completing the square*" in a quadratic equation, (called the *Hindoo* method), which is not so often used as it ought to be, for it has decidedly the advantage of the common method in many cases, as will be seen from the subjoined Examples.

The Rule is, when an equation is in the form $ax^2+bx=c$, (where b and c may be either positive or negative) multiply both sides by $4a$, that is, 4 times the coefficient of x^2, then add to both sides b^2, that is, the square of the coefficient of x, and the left hand side will be a "complete square", without introducing *fractions*, as in the other method.

Ex. 1. If $3x^2 + 2x = 85$, find x.

Multiply by 4×3 or 12, $36x^2 + 24x = 1020$,

add 2^2, or 4, $36x^2 + 24x + 4 = 1024$,

extract root, $6x + 2 = \pm 32$,

$$6x = \pm32 - 2 = 30, \text{ or } -34,$$
$$\therefore x = 5, \text{ or } -5\tfrac{2}{3}.$$

Ex. 2. If $5x^2 - 9x + 2\frac{1}{4} = 0$, find x.

Transposing, $5x^2 - 9x = -2\frac{1}{4}$,

multiply by 4×5, or 20, $100x^2 - 180x = -45$,

add 9^2, or 81, $100x^2 - 180x + 81 = 81 - 45 = 36$,

extract root, $10x - 9 = \pm 6$,

$$10x = 9 \pm 6 = 15, \text{ or } 3,$$
$$\therefore x = \frac{15}{10}, \text{ or } \frac{3}{10},$$
$$= 1\tfrac{1}{2}, \text{ or } \frac{3}{10}.$$

EXERCISES. W.

Find the values of x in each of the following equations:—

(1) $x^2 = 3x + 10$.

(2) $x^2 = 5x - 4$.

(3) $x^2 - 9x = x - 16$.

(4) $x^2 - 14x = 120$.

(5) $12x - 20 = x^2$.

(6) $4x - x^2 = 4$.

(7) $7x - x^2 = 6$.

(8) $x = x^2 - 30$.

(9) $x^2 + \frac{x}{2} = 3$.

(10) $x^2 - \frac{3x}{2} = 27$.

(11) $x^2 + \frac{9x}{2} = 63$.

(12) $9x - 5x^2 = 2\frac{1}{4}$.

(13) $7x + 3x^2 = 6$.

(14) $\frac{x^2}{3} + \frac{3x}{2} = 21$.

(15) $x^2 - \frac{x}{3} = 34$.

(16) $11x^2 - 9x = 11\frac{1}{4}$.

(17) $3x^2 - 5x + 2 = 0$.

(18) $\frac{1}{2}x^2 - \frac{1}{3}x - 2\frac{2}{3} = 0$.

(19) $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{3}{8} = 8$.

(20) $\frac{3}{4}x^2 - \frac{2}{3}x = 1\frac{2}{3}$.

(21) $5(x^2+1) - 3(x-1) = 22$.

(22) $x^2 - 4 = 16 - (x-2)^2$.

(23) $3(x-2)^2 - 3 = 8(x+2)$.

(24) $\frac{3}{4}(x^2 - 3) = \frac{1}{8}(x - 3)$.

(25) $3(2-x) + 2(3-x) = 2(4+3x^2)$.

(26) $x^2 + (x+1)^2 = \frac{13}{6}x(x+1)$.

(27) $4(x-1) - \frac{x-1}{2x} = 3\frac{3}{4}$.

(28) $\frac{6}{x+1} + \frac{2}{x} = 3$.

(29) $\frac{80}{x+4} = \frac{80}{x} - 1$.

(30) $\frac{1}{x-1} - \frac{1}{x+3} = \frac{1}{35}$.

(31) $\frac{x+2}{x-1} - \frac{4-x}{2x} = 2\frac{1}{3}$.

(32) $\frac{4x}{5-x} - \frac{4(5-x)}{x} = 15$.

(33) $\frac{3x-7}{x} = 3\frac{1}{4} - \frac{4(x-2\frac{1}{4})}{x+5}$.

(34) $\frac{4x-3}{3x-7} - \frac{2x-3}{x-1} = 3$.

(35) $\frac{2+x}{7-x} + \frac{2-x}{2+x} = 2\frac{3}{10}$.

(36) $\frac{3x-5}{3x+5} + \frac{135}{176} = \frac{3x+5}{3x-5}$.

(37) $\frac{3x+2}{3x-2} + \frac{3x-2}{3x+2} = \frac{15x+11}{3x+2}$.

(38) $\frac{3}{5-x} + \frac{2}{4-x} = \frac{8}{x+2}$.

(39) $\frac{2x+3}{10-x} = \frac{2x}{25-3x} - 6\frac{1}{2}$.

(40) $\frac{x+8}{x+12} + \frac{5}{x+4} = \frac{3x+14}{3x+8}$.

71. When there are *two* equations and two unknown quantities, and the rules have been applied, as in *Simple Equations*, for reducing them to *one* of one unknown quantity, the resulting equation will sometimes be *quadratic;* and, if the unknown quantity be found from this equation by one of the methods just laid down, the other unknown quantity may be found by *substituting* the value of the former in one of the proposed equations, and solving the resulting equation, which will then contain only one unknown quantity.

Ex. 1. If $\left.\begin{aligned} 2x-8 &= x-y, \\ \text{and } xy-y &= 2x+2, \end{aligned}\right\}$ find x and y.

From 1st equation $2x-x=8-y$,

$$\therefore\ x=8-y, \text{ or } y=8-x;$$

substituting this value of y in the 2nd equation,

$$x(8-x)-(8-x)=2x+2,$$

$$8x-x^2-8+x=2x+2,$$

$$8x+x-2x=x^2+10,$$

$$x^2-7x=-10,$$

$$x^2-7x+\left(\frac{7}{2}\right)^2=\frac{49}{4}-10=\frac{9}{4},$$

$$x-\frac{7}{2}=\pm\frac{3}{2},$$

$$\therefore\ x=\frac{7\pm3}{2}=5, \text{ or } 2.$$

And $y=8-x=8-5$, or $8-2$, $=3$, or 6.

Ex. 2. If $\left.\begin{aligned} 2x^2-3xy &= 2, \\ \text{and } 3x+\ 2y &= 8, \end{aligned}\right\}$ find x and y.

Multiply 1st equation by 2, $\left.\begin{aligned} 4x^2-6xy &= 4, \\ 9x^2+6xy &= 24x, \end{aligned}\right\}$
...... 2nd ...... by $3x$,

adding, $13x^2=4+24x$,

transposing, $13x^2-24x=4$,

$$x^2-\frac{24}{13}x=\frac{4}{13},$$

$$x^2-\frac{24}{13}x+\left(\frac{12}{13}\right)^2=\frac{4}{13}+\frac{144}{(13)^2}=\frac{52+144}{(13)^2}=\frac{196}{(13)^2},$$

extracting root, $x-\frac{12}{13}=\pm\frac{14}{13}$,

$$\therefore\ x=\frac{12\pm14}{13}=\frac{26}{13}, \text{ or } \frac{-2}{13}, =2, \text{ or } \frac{-2}{13}.$$

And $2y=8-3x=8-6$, or $8+\frac{6}{13}$, $=2$, or $8\frac{6}{13}$, $\therefore y=1$, or $4\frac{3}{13}$.

It is not suited to the design of the present work to go on to the solution of more complex and difficult equations, which require the application of some *artifice* suggested by the particular form in which each individual equation is given, and for which no general rule can be stated. The *aspiring* student is referred to *Wood's Algebra, Lund's Edition*, in the Appendix to which this interesting part of the subject is treated at great length.

EXERCISES. X.

Find the values of x and y in the following equations:

(1) $\left.\begin{array}{r} x-2y=0, \\ 3x^2-2y^2=40, \end{array}\right\}$

(2) $\left.\begin{array}{r} 5xy-3y^2=100, \\ 5x-4y=0, \end{array}\right\}$

(3) $\left.\begin{array}{r} \frac{1}{2}x+2y=0, \\ \frac{1}{3}x^2-3y^2=21, \end{array}\right\}$

(4) $\left.\begin{array}{r} 6(x-y)=27, \\ xy=28, \end{array}\right\}$

(5) $\left.\begin{array}{r} 3(x+y)=2\frac{1}{4}, \\ 8xy=1, \end{array}\right\}$

(6) $\left.\begin{array}{r} 2x+3y=11, \\ x^2+xy=4, \end{array}\right\}$

(7) $\left.\begin{array}{r} y-x=2, \\ 10x+y=3xy, \end{array}\right\}$

(8) $\left.\begin{array}{r} 2x-3y=1, \\ 2x^2+xy-5y^2=20, \end{array}\right\}$

(9) $\left.\begin{array}{r} 5x-2y=4, \\ 3x^2+4xy=36, \end{array}\right\}$

(10) $\left.\begin{array}{r} x+y=5, \\ x^2+y^2=13, \end{array}\right\}$

(11) $\left.\begin{array}{r} 3x+2y=14, \\ 2x^2+3y^2=56, \end{array}\right\}$

(12) $\left.\begin{array}{r} \frac{4x}{5y}=\frac{14}{15}, \\ x^2+y^2-xy-7y=1. \end{array}\right\}$

PROBLEMS

DEPENDING UPON THE SOLUTION OF QUADRATIC EQUATIONS.

PROB. 1. FIND the number which multiplied by the half of itself produces 50.

Let x be the number required,

$$\text{then } \frac{x}{2}=\text{its half,}$$

$$\therefore\ x \cdot \frac{x}{2} = 50, \text{ by the question,}$$

$$\frac{x^2}{2} = 50,$$

$$x^2 = 100,$$

$\therefore\ x = \pm 10$, the number required; both $+10$, and -10, satisfying the Problem.

PROB. 2. The sum of £4. 10*s.*, is equally divided among a certain number of persons, and each receives as many half-crowns as there are persons altogether. What is the number?

Let x be the number of persons; then each person receives x half-crowns, or $x \times 2\frac{1}{2}$ shillings; and $\therefore$ the sum received by all together $= x \times x \times 2\frac{1}{2}$ shillings; but the whole sum is 90*s.*

$$\therefore\ x \times x \times 2\tfrac{1}{2} = 90, \text{ by the question,}$$

$$x^2 = \frac{90}{2\frac{1}{2}},$$

$$x^2 = \frac{180}{5} = 36,$$

$$\therefore\ x = \pm 6;$$

$\therefore$ the number required is 6, the *negative* value having no meaning in this Problem.

PROB. 3. A person bought a lot of pigs for £4. 16*s.* which he sold again at 13*s.* 6*d.* per head, and gained by the whole as much as one pig cost him. What number did he buy?

Let x be the number required,

then, $\because$ £4. 16*s.* is 96*s.*, the *cost* price is $\frac{96}{x}$ per head, in shillings,

and $13\frac{1}{2} \times x =$ what he *sold* the whole for, in shillings,

$\therefore\ 13\frac{1}{2} \times x - 96 =$ his profit, ...

Hence, by the question, $13\frac{1}{2} \times x - 96 = \frac{96}{x}$,

multiply by $2x$, $\quad 27x^2 - 192x = 192$,

divide by 3, $\quad 9x^2 - 64x = 64$,

complete the square, $x^2 - \frac{64}{9}x + \left(\frac{32}{9}\right)^2 = \frac{64}{9} + \frac{1024}{81} = \frac{1600}{81}$,

$$x - \frac{32}{9} = \pm\frac{40}{9},$$

$$\therefore\ x = \frac{32 \pm 40}{9} = \frac{72}{9},\ \text{or} -\frac{8}{9},$$

$$= 8,\ \text{or} -\frac{8}{9}.$$

$\therefore$ the number required is 8.

Prob. 4. A gardener, who had no knowledge of Arithmetic, undertook to plant a certain number of trees at equal distances apart, and in the form of a square. In the first attempt, when he had finished his square, he had 11 trees to spare. He then added one of these to each row, as far as they would go, and found that he wanted 24 trees more to complete his square. How many trees were there?

Let x be the number in the *side* of the first square, then $x.x$, or x^2 = number of trees in the whole square,

$\therefore\ x^2 + 11$ = all the trees, by the question.

Again, $x + 1$ = number in a side of the second square,

$\therefore\ (x+1)(x+1)$, or $(x+1)^2$ = all the trees in this square completed,

$$\therefore \text{ by the question, } (x+1)^2 - 24 = x^2 + 11,$$

$$x^2 + 2x + 1 - 24 = x^2 + 11,$$

$$2x = 34,$$

$$\therefore\ x = 17, \text{ and } x^2 = 289,$$

$$\therefore \text{ number of trees} = x^2 + 11 = 289 + 11 = 300.$$

Prob. 5. A printer, reckoning the cost of printing a book at so much per page, made the whole book come to £16. It turned out however that the book contained 5 pages more than he reckoned, and an abatement also was made of 2 shillings per page. He received £13. 10*s*. How many pages did the book contain?

Let x be the number of pages in the book,

then $\because$ £16 = 320*s*. the price he first reckoned was $\frac{320}{x}$*s*. per page,

and $\because$ £13. 10*s*. = 270*s*., the price for $\overline{x+5}$ pages,

the price he received was $\frac{270}{x+5}s.$ per page, which was $2s$, less than the former, by the question,

$$\therefore \frac{320}{x} = \frac{270}{x+5} + 2,$$

first divide by 2, $\frac{160}{x} = \frac{135}{x+5} + 1,$

$$160x + 800 = 135x + x^2 + 5x,$$
$$x^2 - 20x = 800,$$
$$x^2 - 20x + 100 = 900,$$
$$x - 10 = \pm 30,$$
$$\therefore x = 10 \pm 30 = 40, \text{ or } -20;$$

$\therefore$ the number of pages is 40; the *negative* value not being applicable to this problem.

PROB. 6. There are 4 consecutive numbers, of which if the first two be taken for the digits of a number, that number is the product of the other two. What are the 4 numbers?

Let x, $x+1$, $x+2$, $x+3$, be the 4 numbers required.

then $10x + \overline{x+1}$ = the number whose digits are x, and $x+1$,

$\therefore$ by the question, $(x+2)(x+3) = 10x + \overline{x+1}$,

$$\text{or } x^2 + 5x + 6 = 11x + 1,$$
$$x^2 - 6x = -5,$$
$$x^2 - 6x + 9 = 9 - 5 = 4,$$
$$x - 3 = \pm 2,$$
$$\therefore x = 3 \pm 2 = 5, \text{ or } 1.$$

Hence the numbers required are 5, 6, 7, 8, or 1, 2, 3, 4, both of which results satisfy the problem,

$$\because 56 = 7 \times 8, \text{ and } 12 = 3 \times 4.$$

PROB. 7. Twenty persons contribute to send a donation of £2. 8*s*. to the Society for Promoting Christian Knowledge, one half of the whole being furnished in equal portions by the women, and the other half by the men; but each man gave a shilling more than each woman. How many were there of each sex, and what did each person contribute?

Let x be the number of women, and y the contribution of each, in shillings,

$\therefore 20 - x$ = the number of men, and $y + 1$ = contribution of each, in shillings,

hence $xy =$ whole contribution of the women,

and $(20-x)(y+1) =$ .... men,

$$\therefore \text{ by the question, } \left.\begin{aligned} xy &= 24, \\ \text{and } (20-x)(y+1) &= 24, \end{aligned}\right\} \text{ to find } x \text{ and } y.$$

From 2nd equation, $20y + 20 - xy - x = 24$,

substituting from 1st, $20 \times \dfrac{24}{x} + 20 - 24 - x = 24$,

$$\text{or } \frac{480}{x} - x = 28,$$

$$\therefore x^2 + 28x = 480,$$

$$x^2 + 28x + (14)^2 = 480 + 196 = 676,$$

$$\therefore x + 14 = \pm 26,$$

$$\therefore x = \pm 26 - 14 = 12, \text{ or } -40.$$

And $20 - x = 20 - 12 = 8$, or $20 + 40 = 60$.

Also $y = \dfrac{24}{x} = \dfrac{24}{12} = 2$, or $\dfrac{24}{-40}$ *i.e.* $\dfrac{-3}{5}$;

$$\text{and } y + 1 = 3, \text{ or } \frac{2}{5}.$$

$\therefore$ the number of women is 12 contributing 2 shillings each
and .... men .. 8 .... 3 .... ;

the other values of x and y, although furnishing a solution of the *Equations*, not belonging to this *Problem*.

EXERCISES. Y.

1. Find the two consecutive numbers whose product is 156.

2. Find the three consecutive numbers whose sum is equal to the product of the first two.

3. Divide 20 into two such parts, that one is the square of the other.

4. Divide 210 into two such parts, that one is the square of the other.

5. Divide 25 into two such parts, that the *sum* of their squares shall be 313.

6. Divide 30 into two such parts, that the *difference* of their squares shall be 300.

7. The product of two numbers is 144, and if each number be increased by 2, their product will then be 200. What are the numbers?

8. Find the number whose square exceeds the number itself by 156.

9. Find the fraction which is greater than its square by $\frac{1}{4}$.

10. Two trains start at the same time to perform a journey of 156 miles, but one travels a mile an hour faster than the other and reaches the end of its journey just one hour before the other; at what rate did each train travel?

11. A student travelled on a coach 6 miles into the country, and walked back at a rate 5 miles less per hour than that of the coach. He found that he was 50 minutes more in returning than going. What was the speed of the coach?

12. A person distributed £5 in equal portions among a certain number of poor men; and another person did the same, but by giving each man a shilling less, relieved 5 more. What was the number of recipients in each case?

13. A person distributed £36 in equal portions among the poor of a certain place. The next year the same amount was distributed, but the number of recipients was diminished by 6, and consequently each received 1*s*. 8*d*. more than in the year before. What was the number of recipients in each year?

14. Two travellers A and B start at the same time from two places distant 180 miles to meet each other. A travelled 6 miles per day more than B, and B travelled as many miles per day as was equal to twice the number of days before they met. How many miles did each travel per day?

15. The fore-wheel of a coach makes 6 revolutions more than the hind-wheel in going 120 yards; but if the rim of each wheel were increased 1 yard, the fore-wheel would then make only 4 revolutions more than the hind-wheel in the same distance. What is the circumference of each wheel?

16. A person, who can walk forwards four times as fast as he can walk backwards, undertakes to walk a certain distance, and one-fourth of it backwards, in a stated time. He finds that, if his speed per hour backwards were one-fifth of a mile less, he must walk forwards 2 miles an hour faster, to gain his object. What is his speed?

NOTE ON EQUATIONS.

72. The rule so frequently applied *to clear an Equation of fractions* really belongs, in most cases, to common *Arithmetic*, rather than to *Algebra;* that is, *in all cases* where the denominators of the fractions are *arithmetical numbers*. For example, let the given equation be

$$\frac{x}{5}+\frac{x}{4}+\frac{x}{3}-\frac{x}{2}=17, \text{ to find } x.$$

$\because \frac{x}{5}=\frac{1}{5}\cdot x, \quad \frac{x}{4}=\frac{1}{4}\cdot x,$ and so on, the equation becomes

$$\frac{1}{5}x+\frac{1}{4}x+\frac{1}{3}x-\frac{1}{2}x=17,$$

$$\text{or } \left(\frac{1}{5}+\frac{1}{4}+\frac{1}{3}-\frac{1}{2}\right)x=17,$$

$$\therefore x=\frac{17}{\frac{1}{5}+\frac{1}{4}+\frac{1}{3}-\frac{1}{2}},$$

which is the value of x, requiring only to be simplified by a process purely *arithmetical*.

It is not meant that this method is more *easy* than the common one; but by it a clear distinction is kept up between *common Arithmetic* and *Algebra;* and whatever difficulty there may be in such cases, it is manifestly such as the student ought to have mastered before he commenced the subject of *Algebra*.

73. Again, the rule for *clearing an Equation of fractions* may often be *suspended* as follows. Let the given equation be,

$$\frac{6x-4}{21}+\frac{x-2}{5x-6}=\frac{2x}{7}, \text{ to find } x.$$

$$\because \frac{6x-4}{21}=\frac{6x}{21}-\frac{4}{21} \text{ (Arts. 33, 26)}=\frac{2x}{7}-\frac{4}{21}, \text{ (Art. 35)},$$

the equation becomes $\frac{2x}{7}-\frac{4}{21}+\frac{x-2}{5x-6}=\frac{2x}{7},$

erasing and transposing, $\dfrac{x-2}{5x-6}=\dfrac{4}{21}$,

$$21x-42=20x-24,$$

$$\therefore\ x=18.$$

This is a method which will often save much trouble.

74. It may be worth while to give here a method also of solving some equations of *two* unknown quantities, which, although obvious enough, seems to have entirely escaped the notice of writers on Algebra. It applies to such equations especially as those in page 86; thus, taking

Ex. 2. $\left.\begin{aligned}54x-121y&=15,\\36x-\ \ 77y&=21,\end{aligned}\right\}$ to find x and y.

$$\begin{aligned}&\textit{Subtracting}, & 18x-44y&=-6,\\ &\textit{multiply}\text{ by } 2, & 36x-88y&=-12,\\ &\text{from 2nd equation}, & 36x-77y&=\ \ 21,\\ &\text{subtracting}, & 11y&=33,\\ && \therefore\ y&=3.\end{aligned}$$

And $18x=44y-6=132-6=126,\ \therefore\ x=7$.

This method saves all the trouble of finding the *Least Common Multiple* of 54 and 36, and is very simple throughout.

Take another example from p. 87,

(16) $\left.\begin{aligned}101x-24y&=63,\\103x-28y&=29,\end{aligned}\right\}$ to find x and y.

$$\begin{aligned}&\textit{Subtracting}, & 2x-4y&=-34,\\ &\textit{multiply}\text{ by } 6, & 12x-24y&=-204,\\ &\text{but} & 101x-24y&=\ \ 63,\\ &\text{subtracting}, & 89x&=267,\\ && \therefore\ x&=\frac{267}{89}=3.\end{aligned}$$

And $4y=2x+34=40,\ \therefore\ y=10$.

Ex. $\left.\begin{aligned}57x-37y&=86,\\54x-35y&=82,\end{aligned}\right\}$ to find x and y. Ans. $\left.\begin{aligned}x&=8,\\y&=10,\end{aligned}\right\}$

RATIO, PROPORTION, AND VARIATION.

75. DEF. RATIO is the relation which one quantity bears to another in respect of *magnitude*, which relation is measured by the *number of times* the one contains the other, or by the part or parts the one is of the other, according as the one is greater or less than the other. Thus the Ratio of 9 to 3 is 3, because 9 contains 3 *three times;* and the Ratio of 3 to 9 is $\frac{1}{3}$, because 3 is *one third part* of 9.

Hence $\frac{a}{b}$ will always represent the *Ratio* of a to b whatever numbers a and b stand for, because, if $a > b$, $\frac{a}{b}$ expresses the *number of times* a contains b; and, if $a < b$, $\frac{a}{b}$ expresses the *part*, or *parts*, a is of b.

$a : b$ is the abbreviated way of writing '*the Ratio of a to b*'; hence $a : b = \frac{a}{b}$. Similarly $c : d = \frac{c}{d}$. Therefore if $\frac{a}{b} = \frac{c}{d}$, $a : b = c : d$, or the Ratio of a to b is equal to the Ratio of c to d. This equality of two Ratios constitutes what is called a 'PROPORTION'. It is usually written thus

$$a : b :: c : d,$$

and is read '*a is to b as c is to d*'.

Thus, since $\frac{2}{3} = \frac{4}{6}$, $2 : 3 :: 4 : 6$, that is, 2 bears the same relation to 3 in respect of magnitude, which 4 does to 6; and 2, 3, 4, 6, are called *proportionals*.

It will be necessary, therefore, for the student constantly to bear in mind, with respect to *Ratio* and *Proportion*, these two things, viz.

1. That the *measure* of any Ratio $a : b$ is $\frac{a}{b}$.

2. That, if $a : b :: c : d$, then $\frac{a}{b} = \frac{c}{d}$.

For as soon as a *ratio* is converted into a *fraction*, or a *proportion* into an *equation*, then, of course the Rules before given for fractions and equations are immediately applicable.

Ex. 1. Which is greater, the ratio $7 : 4$, or the ratio $8 : 5$?

$$\left.\begin{array}{l} 7 : 4 \text{ is } \textit{measured} \text{ by } \dfrac{7}{4}, \\ 8 : 5 \text{ .... } \dfrac{8}{5}, \end{array}\right\} \begin{array}{l} \therefore 7 : 4 > \text{or} < 8 : 5 \\ \text{according as } \dfrac{7}{4} > \text{or} < \dfrac{8}{5}, \end{array}$$

or, (bringing the two fractions to a common denominator, which does not alter their *value*,) according as $\dfrac{35}{20} >$ or $< \dfrac{32}{20}$,

$$\text{and } \frac{35}{20} > \frac{32}{20}, \left(\text{for } \frac{35}{20} = \frac{32}{20} + \frac{3}{20}\right), \ \therefore 7 : 4 > 8 : 5.$$

[*Exercises* Z, 1...12, p. 125.]

76. If the terms of a ratio be multiplied or divided by the same quantity, the *value* of the ratio is not altered.

For let $a : b$ be any ratio, then

$$a : b = \frac{a}{b} \text{ (Art. 75), and } \frac{a}{b} = \frac{ma}{mb}, \text{ (Art. 34)},$$

$$\therefore a : b = \frac{ma}{mb} = ma : mb.$$

$$\text{Conversely } ma : mb = \frac{ma}{mb} = \frac{a}{b} = a : b.$$

Exs. $2 : 3 = 4 : 6$, $5 : 2 = 15 : 6$, $1 : 5 = 10 : 50$.

[*Exercises* Z, 13...18, p. 125.]

77. If $a : b :: c : d$, shew that $ad = bc$, and the converse.

Since $a : b :: c : d$, $\dfrac{a}{b} = \dfrac{c}{d}$, by definition of *Proportionals*, and multiplying these equal quantities by bd,

$$\frac{abd}{b} = \frac{cbd}{d} \text{ (Art. 38); but } abd = b.ad, \text{ and } cbd = d.bc,$$

$$\therefore \frac{b.ad}{b} = \frac{d.bc}{d}, \text{ or } ad = bc.$$

Conversely, if $ad = bc$, dividing these equal quantities by bd,

$$\frac{ad}{bd} = \frac{bc}{bd}, \text{ or } \frac{a}{b} = \frac{c}{d} \text{ (Art. 35), or } a : b = c : d,$$

$$\therefore a : b :: c : d.$$

Hence, also, if three terms of a proportion be given, the fourth may be found.

For, if $a : b :: c : x$, by what has been proved above

$$ax = bc, \ \therefore x = \frac{bc}{a}.$$

This is the proof of *The Single Rule of Three* in Arithmetic, which teaches how to find the *fourth* term of a *proportion*, when *three* terms are given.

78. If $a : b :: c : d$, shew that $b : a :: d : c$.

Since $a : b :: c : d$, $\frac{a}{b} = \frac{c}{d}$, (Art. 75),

multiply these equal quantities by bd, then $ad = bc$,

$$\text{divide by } ac,\ \frac{ad}{ac} = \frac{bc}{ac},$$

$$\text{or } \frac{d}{c} = \frac{b}{a},\ \text{(Art. 35)},$$

$$\text{or } \frac{b}{a} = \frac{d}{c},$$

$$\therefore b : a :: d : c.$$

79. If $a : b :: c : d$, shew that $a : c :: b : d$.

Since $a : b :: c : d$, $\frac{a}{b} = \frac{c}{d}$, (Art. 75),

$$\text{multiply by } \frac{b}{c},\ \frac{b}{c} \cdot \frac{a}{b} = \frac{b}{c} \cdot \frac{c}{d},\ \text{or } \frac{ab}{bc} = \frac{bc}{cd},$$

$$\text{or } \frac{a}{c} = \frac{b}{d},\ \text{(Art. 35)},$$

$$\therefore a : c :: b : d.$$

80. If $a : b :: c : d$, shew that $a + b : b :: c + d : d$.

Since $a : b :: c : d$, $\frac{a}{b} = \frac{c}{d}$, (Art. 75),

$$\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1,$$

$$\text{or } \frac{a+b}{b} = \frac{c+d}{d},$$

$$\therefore a + b : b :: c + d : d.$$

81. If $a : b :: c : d$, and $c : d :: e : f$, shew that $a : b :: e : f$.

$$\left.\begin{aligned} &\text{Since } a : b :: c : d,\ \frac{a}{b} = \frac{c}{d}, \\ &\text{and} \because c : d :: e : f,\ \frac{c}{d} = \frac{e}{f}, \end{aligned}\right\}\ \text{(Art. 75)},$$

$$\therefore \frac{a}{b} = \frac{e}{f},$$

$$\text{or } a : b :: e : f.$$

82. If $a : b :: c : d$, and $b : e :: d : f$, shew that $a : e :: c : f$.

$$\left.\begin{array}{ll}\text{Since } a : b :: c : d, & \dfrac{a}{b} = \dfrac{c}{d}, \\ \text{and } \because b : e :: d : f, & \dfrac{b}{e} = \dfrac{d}{f},\end{array}\right\} \text{(Art. 75)},$$

$$\therefore \frac{a}{b} \times \frac{b}{e} = \frac{c}{d} \times \frac{d}{f},$$

$$\text{or } \frac{ab}{be} = \frac{cd}{df}, \text{ or } \frac{a}{e} = \frac{c}{f}, \text{ (Art. 35)},$$

$$\therefore a : e :: c : f.$$

[*Exercises* Z, 19...28, p. 126.]

83. *To shew that if quantities be proportional according to the Algebraical definition, they are proportional according to the Geometrical definition*.*

Let a, b, c, d represent four quantities in proportion according to the *Algebraical* definition; then we have

$$\frac{a}{b} = \frac{c}{d}, \text{ (Art. 75)},$$

$$\therefore \frac{m}{n} \cdot \frac{a}{b} = \frac{m}{n} \cdot \frac{c}{d}, \text{ multiplying equal quan-}$$

tities by the same quantity, $\dfrac{m}{n}$,

$$\text{or } \frac{ma}{nb} = \frac{mc}{nd}, \text{ (Art. 40)},$$

from which it follows, by the nature of fractions, that if $ma > nb$, then $mc > nd$; if $ma = nb$, then $mc = nd$; and if $ma < nb$, then $mc < nd$. And ma, mc, are any *equimultiples whatever* of the 1st and 3rd quantities; and nb, nd are any *equimultiples whatever* of the 2nd and 4th, since m and n are *any* whole numbers *whatever*. Therefore a, b, c, d are proportionals also according to the *Geometrical Definition*.

84. VARIATION. DEF. *Variable* or *Varying* quantities are such as admit of various values in the same computation. *Constant* or *invariable* quantities have only one fixed value.

One quantity is said to '*vary directly*' as another, when the two quantities depend upon each other *in such manner*, that if one be changed, the other is changed *in the same proportion*.

* For the Geometrical Definition of Proportion see Euclid, Book v. Def. 5.

Thus, let A and B be two variable quantities mutually dependent upon each other, *in such a way*, that if A is changed to any other value a, B becomes b, these changes being such that $A : a :: B : b$; then A is said to *vary directly* as B.

For example, if a man agrees to work for a certain sum per hour, the amount of his wages will *vary directly* as the number of hours he works; for as the hours increase or decrease, so also will the wages, and *in the same proportion*.

N.B. It often happens that two quantities are mutually dependent upon each other, and yet do not '*vary*' as each other. They may increase or decrease together, and yet one shall not '*vary*' as the other, because the changes in the two are not *proportional*. For example, the side and area of a square are mutually dependent upon each other, so that the one cannot be changed without the other being changed, but the changes are not *proportional*, that is, when the side is *doubled*, the area is not doubled, it is *quadrupled*—when the side is *trebled*, the area becomes *nine times* its former value, and so on.

When it is simply stated that one quantity '*varies*' as another, it is always meant that the one 'varies *directly*' as the other, in the sense above given. The symbol $\propto$ is used to signify that the quantities between which it is placed '*vary*' as each other.

Ex. Given that $y \propto x$, and when $x = 2$, $y = 20$, state the resulting *proportion*.

Here, when y is changed to 20, x is changed to 2, and $y \propto x$,

$$\therefore\ y : 20 :: x : 2, \text{ or } y : x :: 20 : 2, \text{ (Art. 79)},$$
$$\text{or } y : x :: 10 : 1, \text{ (Art. 76)}.$$

85. Def. One quantity is said to '*vary inversely*' as another, when the one cannot be changed in any manner, without the *reciprocal** of the other being changed *in the same proportion*.

A varies inversely as B, $\left(\text{which is written thus } A \propto \frac{1}{B}\right)$, if, when A is changed to a, B be changed to b, such that $A : a :: \frac{1}{B} : \frac{1}{b}$, or, multiplying the last two terms by Bb, (Art. 76), $A : a :: b : B$.

* By '*reciprocal*' of a quantity is meant $\frac{1}{\textit{that quantity}}$. Thus the '*reciprocal* of a is $\frac{1}{a}$, whatever quantity a stands for.

For example, if a letter-carrier has a fixed route, the time in which he will finish his work *varies inversely* as his speed. If he *double* his speed, he will go in *half* the time: and similarly, however he may alter his speed, (provided it be *uniform* throughout the journey, which is here supposed) the '*reciprocal*' of the time will manifestly be altered *in the same proportion*.

Ex. Given that y varies inversely as x, $\left(y \propto \frac{1}{x}\right)$, and when $x = 3$, $y = 1$, find the resulting proportion.

Here $y : 1 :: \frac{1}{x} : \frac{1}{3}$, or $y : \frac{1}{x} :: 1 : \frac{1}{3}$, (Art. 79),

or $y : \frac{1}{x} :: 3 : 1$, (Art. 76).

86. Def. One quantity is said to '*vary as two others jointly*', if, when the first is changed in any manner, the *product* of the two others is changed *in the same proportion*.

A varies as B and C jointly, (which is written $A \propto BC$), if, when A is changed to a, BC becomes bc, such that $A : a :: BC : bc$.

For example, the wages to be received by a workman will vary as the number of days he has worked and the wages per day *jointly*, for if either the number of days or the wages per day be doubled, trebled, &c. so as to double or treble, &c. their *product*, the whole wages to be received for the work will likewise be doubled, or trebled, &c., that is, altered *in the same proportion*.

Ex. Given that $z \propto xy$, and when $x = 1$, and $y = 2$, $z = 20$, find the resulting *proportion*.

Here $z : 20 :: xy : 1 \times 2$, $\therefore z : xy :: 20 : 2$, (Art. 79),

or $z : xy :: 10 : 1$, (Art. 76).

87. Any *variation* may be converted into an equivalent *equation* when two corresponding values of the variable quantities are known.

For, if $A \propto B$, and a, b are known corresponding values of A and B, then

$$A : a :: B : b, \text{ by definition,}$$

$$\therefore Ab = aB, \text{ (Art. 77),}$$

$$\text{or } A = \frac{a}{b} \cdot B.$$

Ex. Given $y \propto x$, and when $x = 1$, $y = 3$, find the *equation* between x and y.

Here $y : 3 :: x : 1$, $\therefore y = 3x$. (Art. 77)

N.B. The most ready method of treating *variations* is in general to convert them into *equations*. For since $\frac{A}{B} = \frac{a}{b}$, always, when $A \propto B$, that is, A and B cannot change value without retaining the same *ratio*, which is, therefore, in each case a fixed *invariable* quantity, it is usual to express that quantity by some assumed letter as m, n, or p. Thus, if $A \propto B$, then $\frac{A}{B} = m$, or $A = mB$; where $m = \frac{a}{b}$. But if, in the same computation, there occurs another variation, as $C \propto D$, we cannot then say $C = mD$, because although $\frac{C}{D} =$ a fixed invariable quantity, it may not be the *same* quantity as in the other variation. So that we should write $C = nD$.

Ex. Given that $y \propto$ the sum of two quantities, one of which varies as x and the other as x^2, find the corresponding equation.

Here, $\because$ one part $\propto x$, this $= mx$, and the other ... $\propto x^2$, ... $= nx^2$, } m and n being invariable,

$$\therefore y = mx + nx^2.$$

The invariable quantities m and n can only be found when we know two pairs of corresponding values of x and y.

[*Exercises* Z, 29...32.]

EXERCISES. Z.

Find the value, or *measure*, of each of the following *Ratios*:—

(1) $3a : 15a$.	(7) $a^2pc : 3acx$.
(2) $2x : 10x^2$.	(8) $3x^3y^2 : 12x^2y^3$.
(3) $ax : bx$.	(9) $ac + bc : c^2$.
(4) $abc : bc$.	(10) $2ax + x^2 : mx$.
(5) $axy : 2x$.	(11) $1 - x^2 : 1 - x$.
(6) $3abx : 2a^2x$.	(12) $a^2 - b^2 : a + b$.

Simplify each of the following *Ratios*:—

(13) $5ax : 4x$.	(17) $\frac{7axy}{1\times2\times3} : \frac{5ay^2}{2\times3\times4}$.
(14) $16xy : 20x^2$.	
(15) $\frac{1}{2}ax : \frac{3}{4}bx$.	(18) $\frac{n(n-1)}{1\times2}ax^3 : na^2x^2$.
(16) $2x^2y : \frac{1}{4}x^3$.	

(19) Which is the greater $15 : 16$, or $16 : 17$?

(20) Which is the greater $2ax : 3by$, or $3a : 2b$, when $x : y :: 2 : 1$?

(21) If $a : b :: c : d$, shew that $2a : 3b :: 2c : 3d$.

(22) If $a : b :: b : c$, shew that $a : c :: a^2 : b^2$.

(23) Convert the proportion $a : a+x :: a-x : b$ into an *equation*.

(24) Convert $x : y :: y : 2a-x$ into an *equation*.

(25) If $a+x : a-x :: 11 : 7$, find the value of $a : x$.

(26) Find two numbers in the ratio of $2 : 3$, and the sum of which : their product :: $5 : 12$.

(27) The 1st, 3rd, and 4th terms of a proportion are ax, $3cx$, and $\dfrac{6bcy}{a}$, what is the 2nd term?

(28) There are two numbers in the ratio $3 : 4$, and if each of them be increased by 5, the resulting numbers are in the ratio $4 : 5$. What are the numbers?

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(29) If $y \propto x$, and when $x=2$, $y=4a$, find the *equation* between x and y.

(30) If $y \propto \dfrac{1}{x}$, and when $x=\frac{1}{2}$, $y=8$, find the *equation* between x and y.

(31) If $1+x \propto 1-x$, shew that $1+x^2 \propto x$.

(32) If $2x+3y \propto 4x+5y$, shew that $x \propto y$.

ARITHMETICAL PROGRESSION.

88. Def. A series of quantities are in *Arithmetical Progression*, when, taken in order, they go on, from the first to the last, either increasing or decreasing *by the same fixed quantity*, called the '*Common Difference*'.

Thus 1, 3, 5, 7, 9, 11, &c. are in *Arith. Prog.*, because each quantity is greater than the one preceding by the *common difference* 2.

So also 20, 19, 18, 17, &c. are in *Arith. Prog.*, because each quantity is less than the one preceding by the *common difference* 1.

Again $2x$, $4x$, $6x$, $8x$, &c. are in *Arith. Prog.*, the *common difference* being $2x$.

The *general form* of a series in *Arith. Prog.* is either

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad \&c.$$
$$\text{or } a, \quad a-d, \quad a-2d, \quad a-3d, \quad \&c.;$$

in the former the quantities go on regularly *increasing*, and in the latter *decreasing*, by the fixed *common difference d.*

Quʸ. Are 1, 3, 4, 7, 8, &c. in *Arith. Prog.*? No, because $3-1=2$, and $4-3=1$, so that the quantities do not increase by the same quantity, *i. e.* by a *common difference.*

Quʸ. Are 1, 5, 9, 13, 17, &c. in *Arith. Prog.*? Yes, because $5-1=4$, $9-5=4$, $13-9=4$, $17-13=4$, &c. shewing that the quantities increase by a *common difference* 4.

89. In the general series

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad \&c.$$

a is called the 1st *term* of the series, or progression, $a+d$ the 2nd *term*, $a+2d$ the 3rd, and so on. Hence $a+(n-1)d$ is the n^{th} *term*, where n stands for any whole number; for,

$$\text{if } n=1, \quad \text{1st term} = a+(1-1)d = a,$$
$$\text{..} \quad n=2, \quad \text{2nd term} = a+(2-1)d = a+d,$$
$$\text{..} \quad n=3, \quad \text{3rd term} = a+(3-1)d = a+2d;$$

and so on; so that $a+(n-1)d$ truly represents the n^{th} term, whatever *number* n be.

Similarly, $a-(n-1)d$ represents the n^{th} term of the *decreasing* series a, $a-d$, $a-2d$, &c.

90. In any series of terms in *Arith. Prog.* we can find any proposed term independently of the rest, if we know the first term and the *common difference.* For a and d being known, $a+(n-1)d$ is known for any given value of n.

Ex. 1. Find the 50th term in the series 1, 5, 9, 13, 17, &c.

Here $a=1$, $d=4$, and $n=50$, therefore substituting these values in $a+(n-1)d$,

$$\text{the term required} = 1+(50-1)\times 4 = 1+200-4 = 197.$$

[*Exercises Za*, 1...3, p. 131.]

91. It is plain that we could find the *sum* of any number of quantities in *Arith. Prog.* by adding them together in the ordinary way; but when the number of terms is large this method would be found inconvenient. The following Rule will give us the sum more readily in all such cases:—

RULE. *To find the sum of a series of quantities in Arith. Prog., add together the first and last terms, and multiply half this sum by the number of terms, or the whole sum by half the number of terms, if more convenient.*

Thus to find the sum of the first 5 terms of the series 1, 5, 9, 13, 17, &c.:

1 is the 1st term, 17 is the last, their sum is 18, half this is 9, which multiplied by 5, the number of terms, gives 45, for the sum of the series. That this is correct, appears by taking the sum $1+5+9+13+17$.

If it were required to find the sum of 100 terms of the same series, we must first find the last term, that is, the 100th, this is

$$1+(100-1)\times 4, \text{ by Art. } 90,$$
$$\text{or } 1+400-4, \text{ that is, } 397.$$

$$\text{Then the sum required} = \tfrac{1}{2}(1+397)\times 100.$$
$$= 199\times 100 = 19900.$$

92. *To prove the Rule generally.*

Since $a,\ a+d,\ a+2d,\ a+3d,\ \ldots l$, where l stands for the last term, will represent any series of quantities increasing in *Arith. Prog.* by the common difference d, let s be the sum of the quantities, then

$$s = a + \overline{a+d} + \overline{a+2d} + \overline{a+3d} + \&c.\ldots + l.$$

Now since the terms go on regularly increasing by the quantity d, the term next before l, will be $l-d$, and the term before that $l-2d$, and so on; therefore reversing the series which cannot alter the sum, we have also

$$s = l + \overline{l-d} + \overline{l-2d} + \&c. + \ldots\ldots + \overline{a+d} + a,$$

adding this to the former, we have

$2s = \overline{a+l} + \overline{a+l} + \overline{a+l} + \&c.$ $\overline{a+l}$ being repeated as many times as there are terms in the series;

$\therefore\ 2s = n$ times $\overline{a+l}$, or $n\times(a+l)$, if n be the number of terms, and

$$\therefore\ s = \tfrac{1}{2}n(a+l).$$

Also, if the series be *decreasing*, the same result will be obtained, merely changing the sign of d in the above operation from + to −, and from − to +.

[*Exercises Za*, 4...14, p. 131.]

93. Having given two quantities a and b, find another, x, so that a, x, b, shall be in *Arith. Prog.* (The middle quantity, x, is called '*the Arithmetic Mean*' between a and b.)

Since a, x, b, are in *Arith. Prog.* by the supposition,

$x-a=$ the *Com. Diff.* Also $b-x=$ the *Com. Diff.*

$$\therefore\ x-a=b-x,$$

$$\text{transposing,}\quad 2x=a+b,$$

$$\therefore\ x=\frac{a+b}{2}.$$

Hence it appears, that *the Arithmetic Mean* between any two quantities is half the *sum* of the quantities.

Ex. 1. The *Arith. Mean* between 6 and 20 is $\frac{1}{2}(6+20)$, or 13; that is, 6, 13, 20 are in *Arith. Prog.*, as they plainly are.

Ex. 2. The *Arith. Mean* between $a+b$, and $a-b$, is $\frac{1}{2}(a+b+a-b)$, or a, that is,

$a+b$, a, $a-b$, are in *Arith. Prog.*

[*Exercises Za*, 15...17, p. 132.]

94. Having given two quantities a and b, find *two* others x and y, such that a, x, y, b, shall be in *Arith. Prog.* (This is called inserting two *Arith. Means* between a and b.)

Since a, x, y, b are in *Arith. Prog.* by the supposition,

$$\left.\begin{aligned}\text{by Art. 93, }x&=\frac{a+y}{2},\\ \text{and }y&=\frac{x+b}{2},\end{aligned}\right\}\ \text{two simple equations for finding } x \text{ and } y.$$

$$\text{From 1st}\quad 2x=a+y,\quad\text{but } y=\frac{x+b}{2},$$

$$\therefore\ 2x=a+\frac{x+b}{2},$$

$$4x=2a+x+b,$$

$$3x=2a+b,$$

$$\therefore\ x=\frac{2a+b}{3}$$

$$\text{Also } y=2x-a=\frac{4a+2b}{3}-a=\frac{a+2b}{3}.$$

Hence a, $\frac{2a+b}{3}$, $\frac{a+2b}{3}$, b, are in *Arith. Prog.*

Verification. $\frac{2a+b}{3}-a=\frac{b-a}{3}$, $\frac{a+2b}{3}-\frac{2a+b}{3}=\frac{b-a}{3}$,
and $b-\frac{a+2b}{3}=\frac{b-a}{3}$;

shewing that the quantities a, x, y, b, increase by a *common difference* $\frac{b-a}{3}$, that is, are in *Arith. Prog.*

95. The same thing may be done more easily in another way, thus: Let the quantities be a, $a+x$, $a+2x$, b, where x is the *common difference* which remains to be found.

Here the *com. diff.* $x=b-(a+2x)$,

$$\text{or } x=b-a-2x,$$

$$3x=b-a,$$

$$\therefore\ x=\frac{b-a}{3}.$$

And the means $a+x$, $a+2x$, are $\therefore\ a+\frac{b-a}{3}$, $a+\frac{2(b-a)}{3}$,

$$\text{or } \frac{2a+b}{3},\ \frac{a+2b}{3}, \text{ as before.}$$

By the latter method *any number* of *Arith. Means* may be inserted between two quantities.

Ex. 1. Find the *Arith. Mean* between $\frac{1}{4}$, and $\frac{1}{2}$.

$$\text{The required Mean}=\frac{1}{2}\left(\frac{1}{4}+\frac{1}{2}\right)=\frac{1}{2}\times\frac{3}{4}=\frac{3}{8},$$

that is, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, are in *Arith. Prog.*

Ex. 2. Insert *two* Arith. Means between $\frac{1}{3}$ and $\frac{11}{6}$.

Let x be the unknown *com. diff.*; then the series is

$$\frac{1}{3},\ \frac{1}{3}+x,\ \frac{1}{3}+2x,\ \frac{11}{6},$$

$$\therefore \frac{11}{6} - \left(\frac{1}{3} + 2x\right) = com.\ diff. = x,$$

$$\frac{9}{6} - 2x = x,$$

$$3x = \frac{9}{6}, \therefore x = \frac{1}{2}.$$

$\therefore$ the required means are $\frac{1}{3} + \frac{1}{2}$, and $\frac{1}{3} + 1$,

that is, $\frac{5}{6}$, and $1\frac{1}{3}$.

Hence $\frac{1}{3}$, $\frac{5}{6}$, $1\frac{1}{3}$, $\frac{11}{6}$, are in *Arith. Prog.*

[*Exercises Za,* 18...25.]

EXERCISES. *Za.*

Find the 15th, and the 20th, terms in each of the following series:—

(1) 1, 6, 11, &c.
(2) 16, 15, 14, &c.
(3) $\frac{1}{3}$, $\frac{2}{3}$, 1, &c.

Find the sum of 20 terms of each of the following series:—

(4) 1, 3, 5, 7, &c.
(5) 5, 8, 11, 14, &c.
(6) 100, 110, 120, &c.
(7) 100, 97, 94, &c.
(8) 15, 11, 7, &c.
(9) $\frac{1}{2}$, $\frac{3}{4}$, 1, &c.
(10) 13, $12\frac{2}{3}$, $12\frac{1}{3}$, &c.

(11) How many strokes does a clock make in 12 hours, which strikes the hours only? Calculate this without writing.

(12) If a labourer were hired for a year to receive a *farthing* the first day, a halfpenny the second, three farthings the third, and so on, excluding Sundays; what would his wages amount to for the year? And what would he receive for the 25th week?

(13) A certain debt was discharged in 25 weeks, by paying 2 shillings the 1st week, 5 shillings the 2nd, 8 shillings the 3rd, and so on. What was the amount of the debt?

(14) How far does a person travel in gathering up 200 stones placed in a straight line at intervals of 2 feet from each other; supposing that he fetches each stone singly and deposits it in a basket, which is in the same line produced 20 yards distant from the nearest stone, and that he starts from the basket?

(15) Find the *Arith. Mean* between $\frac{1}{4}$ and $\frac{1}{9}$.

(16) Find the *Arith. Mean* between $1 + x$, and $1 - x$.

(17) Find the *Arith. Mean* between $\frac{a}{2}$ and $\frac{b}{2}$.

(18) Insert 2 *Arith. Means* between 5 and 14.

(19) Insert 3 *Arith. Means* between 1 and 3.

(20) Insert 4 *Arith. Means* between 100 and 80.

(21) There is a series of terms in *Arith. Prog.* of which the sum of the first two terms is $2\frac{1}{2}$, and the 4th term is $2\frac{1}{2}$. What is the series?

(22) In the series 1, 3, 5, 7, 9, &c. the sum of 2 terms is 2^2, the sum of 3 terms is 3^2, of 4 terms 4^2, and so on; *prove* that this is true *generally*, viz. that the sum of n terms is n^2.

(23) The first and last of 40 numbers in *Arith. Prog.* are $1\frac{1}{3}$, and $1\frac{2}{3}$; what are the intervening terms? And what is the sum of the whole series?

(24) An insolvent tradesman agreed to pay a certain debt by weekly instalments, beginning with 5*s.* and increasing by 3*s.* every week. His last payment was £15. 2*s.* For how many weeks did he pay, and what was the whole amount of his debt?

(25) It is shewn in treatises on Dynamics, that a heavy body, falling from rest and unobstructed, passes through a space of $16\frac{1}{10}$ feet nearly in the 1st second of time, but afterwards in each succeeding second $32\frac{1}{5}$ feet more than in the second immediately preceding. Now a heavy body fell from

the car of a balloon and it was ascertained to have been exactly 20 seconds before it struck the earth. What was the height of the balloon, supposing the resistance of the air not worth reckoning?

GEOMETRICAL PROGRESSION.

96. Def. A series of quantities are in *Geometrical Progression* when, taken in order, they go on, from first to last, increasing or decreasing in the same fixed *Ratio*, that is, by a common multiplier.

This multiplier is called *the Common Ratio*, and may be either whole or fractional.

Thus, 1, 2, 4, 8, 16, &c. are in *Geom. Prog.* because each quantity is *twice* as great as the one preceding.

So also 16, 8, 4, 2, 1 are in *Geom. Prog.* because each quantity is *half* as great as the one preceding.

In the first series the *Common Ratio* is 2, in the second $\frac{1}{2}$. In any series of this kind the *Common Ratio* is found by dividing *any term by the one preceding;* and if *every* term divided by the preceding one do not give the *same quotient* the series is *not* in *Geom. Prog.*

Ex. 1. To find the *Common Ratio* in the series 1, 3, 9, 27, &c.

Here *Common Ratio* $= \frac{3}{1}$, or 3.

Ex. 2. To find the *Common Ratio* in the series $1\frac{1}{2}$, 3, 6, 12, &c.

Here the Common Ratio $= \frac{6}{3}$, or 2. In this case it is more convenient to divide the *third* term by the one preceding, than the 2nd by the 1st.

Ex. 3. Are $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{3}{4}$ in *Geom. Prog.;* and if so, what is the Common Ratio?

$$\text{Here } \frac{1}{2} \div \frac{1}{3} = \frac{3}{2}, \text{ and } \frac{3}{4} \div \frac{1}{2} = \frac{3}{2},$$

therefore the quantities *are* in *Geom. Prog.*, and the Common Ratio is $\frac{3}{2}$.

[*Exercises Zb*, 1...9, p. 137.]

97. The *general form* of a series in *Geom. Prog.* is

$$a,\ ar,\ ar^2,\ ar^3,\ \&c.$$

where each term is r times the preceding one, r being either whole or fractional. The n^{th} term is manifestly ar^{n-1}, because

$$\begin{aligned}&\text{the 2nd is } ar^1,\\ &\ldots\ \text{3rd} \ldots ar^2,\\ &\ldots\ \text{4th} \ldots ar^3,\end{aligned}$$

and so on; the index of the power of r being always less by 1 than the number which marks the position of the term.

98. In any series of terms in *Geom. Prog.* we can find any proposed term independently of the rest, if we know the 1st term and the *Common Ratio.* For a and r being known ar^{n-1} is known for any given value of n.

Ex. 1. Find the 8th term in the series 1, 3, 9, 27, &c.

Here $a=1$, $r=3$, and $n=8$, therefore substituting these values in ar^{n-1}, the term required $=1\times 3^7=2187$.

The *sum* of a series of terms in *Geom. Prog.*, like the sum of any other quantities, may be found by adding them together; but, when the number of the terms is large, the following article will furnish a method of summing the series which is more generally applied:—

99. *To find the sum of any number of quantities in Geom. Prog.*

Let $a, b, c, d,$ &c. $k, l,$ be n quantities in *Geom. Prog.*, and let r be the *Com. Ratio;* then, by definition,

$$\begin{aligned}b &= ar,\\ c &= br,\\ d &= cr,\\ \&c. &= \&c.\\ l &= kr,\end{aligned}$$

$$\therefore\ b+c+d+\&c.+l=(a+b+c+\&c.+k)r,$$

or, if s be the sum required, $\because\ b+c+\&c.+l$ is the whole series except the first term, and $a+b+c+\&c.+k$ the whole series except the last term,

$$\begin{aligned}s-a &= (s-l)r,\\ &= rs-rl,\\ \therefore\ (r-1)s &= rl-a,\\ \therefore\ s &= \frac{rl-a}{r-1}.\end{aligned}$$

By substituting in this formula the given values of a, l, and r, for any proposed series, the sum s is found.

Ex. Find the sum of the series 1, 2, 4, 8, &c., 1024.

Here $a = 1$, $l = 1024$, $r = 2$,

$$\therefore\ s = \frac{2 \times 1024 - 1}{2 - 1} = 2047.$$

That this is the correct sum of the series may be verified by actually adding together 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024.

[*Exercises* Zb, 10...13, p. 137.]

100. Having given two quantities a and b, find another x, so that a, x, b, shall be in *Geom. Prog.* (The middle quantity x is called the *Geometric Mean* between a and b.)

Since a, x, b, are in *Geom. Prog.*, by the supposition,

$$\frac{x}{a} = \text{Com. Ratio} = \frac{b}{x},$$

$$\therefore\ x^2 = ab,$$

$$\text{and } x = \sqrt{ab}.$$

Hence it appears that the *Geom. Mean* between any two quantities is the square root of their product.

Ex. 1. The *Geometric Mean* between 16 and 64 is $\sqrt{16 \times 64}$ or $\sqrt{1024}$ or 32; that is, 16, 32, 64 are in *Geom. Prog.*, as they plainly are.

Ex. 2. The *Geom. Mean* between $\frac{a}{b}$, and $\frac{b}{a}$ is $\sqrt{\frac{a}{b} \cdot \frac{b}{a}}$ or $\sqrt{1}$, or 1; that is, $\frac{a}{b}$, 1, $\frac{b}{a}$, are in *Geom. Prog.*

[*Exercises* Zb, 14, 15, p. 137.]

101. Having given two quantities a and b, find two others, x and y, such that a, x, y, b are in *Geom. Prog.* This is called *inserting* two *Geometric Means* between a and b.

Let r be the unknown Com. Ratio,

$$\left.\begin{aligned} \text{then } x &= ar, \\ y &= xr, \\ b &= yr, \end{aligned}\right\}, \text{ by definition.}$$

From 2nd equation $yr = xr^2$, multiplying by r,

$$\therefore\ xr^2 = b.$$

From 1st equation $xr^2 = ar^3$, multiplying by r^2,

$$\therefore\ ar^3 = b,$$

$$\therefore\ r^3 = \frac{b}{a},$$

$$\therefore\ r = \sqrt[3]{\frac{b}{a}}.$$

Hence $x = a\sqrt[3]{\frac{b}{a}}$, and $y = xr = a\left(\sqrt[3]{\frac{b}{a}}\right)^2$.

102. The same thing may be done more easily as follows:—

Let a, ar, ar^2, b, be the quantities, r being the *unknown* Common Ratio, to be found.

Then, the Common Ratio, or r, $= \frac{b}{ar^2}$, (Art. 96,)

$$\therefore\ r^3 = \frac{b}{a}, \text{ multiplying by } r^2,$$

$$\therefore\ r = \sqrt[3]{\frac{b}{a}}.$$

And the means are $a.\sqrt[3]{\frac{b}{a}}$, and $a\left(\sqrt[3]{\frac{b}{a}}\right)^2$, as before.

By this latter method *any number* of *Geometric Means* may be inserted between two given quantities.

Ex. 1. Find the *Geometric Mean* between $\frac{1}{3}$ and $\frac{3}{4}$.

$$\text{Required mean} = \sqrt{\frac{1}{3}\times\frac{3}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}.$$

Ex. 2. Insert two *Geometric Means* between $\frac{1}{9}$ and 3.

Let x be the *Common Ratio;* then the series is

$$\frac{1}{9},\ \frac{1}{9}x,\ \frac{1}{9}x^2,\ 3;$$

$$\therefore\ 3 \div \frac{1}{9}x^2 = \textit{Com. Ratio} = x,$$

$$x^3 = 27,\ \therefore\ x = 3,$$

$$\therefore \text{ the means are } \frac{1}{9}\times 3, \text{ and } \frac{1}{9}\times 3^2,$$

$$\text{that is, } \frac{1}{3}, \text{ and } 1.$$

[*Exercises Zb*, 16...23.]

EXERCISES. *Zb.*

Find the '*Common Ratio*' in each of the following series in *Geom. Prog.*:—

(1) 100, 200, 400, &c.

(2) $2\frac{1}{2}$, 5, 10, &c.

(3) $\frac{1}{3}$, 1, 3, &c.

(4) $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c.

(5) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, &c.

(6) 0·1, 0·01, 0·001, &c.

(7) 1·25, 2·5, 5, &c.

(8) ax, $2a^2x$, $4a^3x$, &c.

(9) $\frac{x}{r}$, $\frac{nx}{r^2}$, $\frac{n^2x}{r^3}$, &c.

(10) The first two terms of a series in *Geom. Prog.* are $\frac{1}{3}$, and 1, what are the next two terms?

(11) The first two terms of a series in *Geom. Prog.* are 125, and 25, what are the 6th and 7th terms?

(12) Find the sum of 5 terms of a series in *Geom. Prog.* of which the 1st term is $\frac{1}{9}$, and the fifth is 9.

(13) Find the sum of 4 terms of a series in *Geom. Prog.* of which the first term is $\frac{16}{27}$, and the 4th is 2.

(14) Find the *Geom. Mean* between 30, and $7\frac{1}{2}$.

(15) Find the *Geom. Mean* between $\frac{1}{3}$, and $\frac{3}{4}$.

(16) Insert two *Geom. Means* between 5, and 320.

(17) Insert two *Geom. Means* between 1, and $\frac{1}{8}$.

(18) Insert three *Geom. Means* between 6, and 486.

(19) Insert three *Geom. Means* between 100, and $2\frac{14}{25}$.

(20) Which is greater the *Arith. Mean*, or the *Geom. Mean*, between 1 and $\frac{1}{9}$? and how much greater?

(21) Are $\frac{x}{y}$, x, xy, in *Geom. Prog.*? and if so, what is the 'Common Ratio'?

(22) A series of terms are in *Geom. Prog.;* the sum of the first two is $1\frac{1}{3}$, and the sum of the next two is 12. Find the series.

(23) A farmer sowed a peck of wheat, and used the whole produce for seed the following year, the produce of this 2nd year again for seed the 3rd year, and the produce of this again for the 4th year. He then sells his stock after harvest, and finds that he has $12656\frac{1}{4}$ bushels to dispose of. Supposing the increase to have been always in the same proportion to the seed sown, what was the annual increase?

MISCELLANEOUS EXERCISES.

FIRST SERIES.

WHAT are the '*terms*' of a quantity in Algebra? What are the *terms* in each of the following quantities?

(1) $2ab + xy$, (2) $mx - nx^2$, (3) $a^2 + b^2 - 2ab$.

(4) What is the *coefficient* or *cofactor* of x in $2x^2$?

(5) What is the *coefficient* of x in $5 + x$?

(6) What is the *coefficient* of 6 in 36?

(7) Do *abc*, *bac*, *cba*, all mean the same thing?

(8) What is the quantity made up of the *factors* 3, x, m^2, and n?

(9) What is the quantity made up of the *factors* ax, $2by$, and z?

Find the value of each of the following quantities when $a = 4$, $b = 6$, and $x = -2$:—

(10) $2ab - x$, (11) $2a^2 - b^2 + x^2$, (12) $ax^2 - bx$.

(13) What is an '*Index*' of a quantity? Which is the *index* in the quantity $3a^4$; and of what is it the index? Express $3a^4$ in *words*.

(14) Find the value of $\frac{2x^2 - ax}{a^3 - x^3}$, when $a = 2$, and $x = -\frac{1}{2}$.

(15) Find the value of $\frac{3a^2b^2c}{4a(3bc - 4a)}$, when $a = 3$, $b = -2$, $c = -4$.

(16) Multiply $2a - 3b$ by $2a + 3b$.

(17) Multiply $3m + n$ by $3m - n$.

(18) Multiply $4a^2 + 3ab + 2b^2$ by $2b^2 - 4a^2 - 3ab$.

(19) Multiply $3a^4$ by $5a^2$, and the product again by $7a^6$.

(20) Multiply a^{n+1} by $2a$, and the product again by a^{n-2}.

(21) Multiply ma^mb^n by nab.

(22) Shew that $\overline{a + b}$ multiplied by $\overline{a - b}$ is equal to $a^2 - b^2$, and express this result in *words*.

Verify the result when $a = 10$, and $b = 8$.

(23) Shew that $\overline{a + b}$ squared is equal to $a^2 + b^2 + 2ab$, and express this result in *words*.

Verify the result when $a = 10$, and $b = 8$.

(24) Divide $15a^2x^7y$ by $3ax^4y$, and the quotient again by $5ax$.

(25) Divide $105a^4x - 140a^3x^2$ by $35a^2x$.

(26) Divide $8ax - 4bcx + 18x^2y$ by $2x$.

(27) Divide $56a^3 + 189$ by $28a^2 - 42a + 63$.

(28) Divide $4b^4 - 16a^4 - 24a^3b - 9a^2b^2$ by $4a^2 + 3ab + 2b^2$.

(29) Divide $9a^{2n} - 6a^nb + b^2$ by $3a^n - b$.

(30) Split up into its simple factors $48a^2bx^3$.

(31) Split up into simple factors $16x^3y^4$ and $28axy$; and find the G.C.M. of the two quantities.

(32) Find the G.C.M. of $9a^3bc$, $2a^2b^3$, and $7abc$.

(33) Find the G.C.M. of $a^2 - x^2$, and $(a + x)^2$.

(34) What are the *factors* of $a^2 + b^2 + 2ab$?

(35) What are the *factors* of $a^2 - b^2$?

(36) What are the *factors* of $4x^2 - a^2$?

(37) What are the *factors* of $16a^2b^2 - 9x^2$?

(38) Find the L.C.M. of a, $2a$, $4a$, and $6a$.

(39) Find the L.C.M. of $4x$, $5y$, $3xy$, and $2y^2$.

(40) Find the L.C.M. of $2a^2x$, $3ax^2$, and $8x^3$.

(41) What does $\dfrac{2a^3 - abc}{a^2 - bc + c^2}$ become, when $a = b = -c$?

(42) Simplify $1 - x + \dfrac{x^2}{1 + x}$.

(43) Simplify $1 - 2x + \dfrac{2x^2}{1 + x}$.

(44) Simplify $\dfrac{5a + 3x}{3} - (a - x)$.

Reduce to lowest terms:—

(45) $\dfrac{abx}{mx - px}$, (46) $\dfrac{x^2 - y^2}{(x - y)^2}$.

(47) One *factor* of $x^2 + 2x - 3$ is $x - 1$, find the other.

(48) One *factor* of $x^2 + 7x + 12$ is $x + 3$, find the other.

(49) *Two factors* of $x^3 - 7x + 6$ are $x - 1$, and $x - 2$, find the other.

(50) Add together $\dfrac{a}{b}$, $\dfrac{3a}{2b}$, and $\dfrac{4a}{5b}$.

(51) Add together $\dfrac{2a}{a - x}$, $\dfrac{x - 2a}{a - x}$, and $\dfrac{a}{a - x}$.

(52) Add together $\dfrac{m}{n - x}$, and $\dfrac{m}{n + x}$.

(53) From $\dfrac{x + 7}{x - 2}$ subtract $\dfrac{x - 3}{x - 2}$.

(54) From $\dfrac{a + 1}{a - 1}$ subtract $\dfrac{a - 1}{a + 1}$.

(55) Multiply $\dfrac{a + 1}{a - 1}$ by $a - 1$.

(56) Multiply $\dfrac{3a}{x} + \dfrac{x}{2}$ by $\dfrac{2a}{3x}$.

(57) Multiply $\dfrac{2a - 6x}{2ax - a^2}$ by $\dfrac{2x - a}{4x}$.

(58) Divide $\dfrac{a}{b} + \dfrac{x}{y}$ by $\dfrac{a}{b} - \dfrac{x}{y}$.

(59) Divide $x - \dfrac{3x}{1 + x}$ by $\dfrac{x(x - 2)}{1 - x}$.

(60) Divide $\frac{3a^2}{2x^2}+\frac{3a}{10x}-\frac{4}{15}$ by $\frac{a}{x}-\frac{1}{3}$.

Simplify

(61) $x-(a-x)$.

(62) $9-2a-(3-5a)$.

(63) $ax-a\{x-a(x-a)\}$.

(64) $1-\frac{1}{2}\{1-\frac{1}{2}(1-4x)\}$.

(65) $1\frac{1}{2}-\frac{1}{5}\{3a+6-\frac{1}{2}(3a-3)\}$.

(66) $\frac{n}{2}(2a-\overline{n-1}\,.\,b)$, when $a=4$, $b=7$, and $n=11$.

(67) What is meant by an '*Equation*'? Is $x+x=2x$ an '*Equation*'? If not, what is it?

(68) What is meant by '*Solving*' an '*Equation*'? What is the '*Solution*' of the equation $x+10=20$?

Solve the following equations:—

(69) $\frac{x}{3}+8=\frac{3x}{4}+3$.

(70) $\frac{1}{2}x-\frac{1}{8}x=5\frac{1}{4}-\frac{1}{2}x$.

(71) $\frac{4x}{5}+\frac{7x}{2}=4\frac{29}{30}-\frac{2x}{3}$.

(72) $3(x+5)=4(51-x)$.

(73) $52-5(2x-1)=27$.

(74) $\frac{3}{7x}-\frac{1}{2}=\frac{5}{14}$.

(75) $\frac{1}{3}(5x-7)-\frac{1}{5}(4x-9)=3\frac{4}{5}$.

(76) $\frac{7x+4}{4}-\frac{4x-1}{7}=5(20-x)$.

(77) $\frac{4x+27}{2}-\frac{9x-12}{2}=24-\frac{9(4x-6)}{10}$.

(78) $\frac{9x+7}{2}-\left(x-\frac{9x-7}{7}\right)=36$.

(79) $\frac{1}{2}x-\frac{1}{3}(x-2)=\frac{1}{4}\{x-\frac{2}{3}(2\frac{1}{2}-x)\}-\frac{1}{3}(x-5)$.

(80) $\frac{2x+3}{4}\div\frac{3x-2}{5}=6\frac{1}{4}$.

(81) $\frac{2x-3}{3x-4}=\frac{4x-5}{6x-7}$.

(82) $\left.\begin{aligned} \frac{x}{3}+\frac{y}{4}&=4, \\ \frac{7y}{2}-11&=\frac{3x}{2}+y, \end{aligned}\right\}$

(83) $\left.\begin{aligned} \frac{1}{4}x+\frac{1}{7}y&=20, \\ 7y+4x&=584, \end{aligned}\right\}$

(84) $\left.\begin{aligned} \frac{1}{2}x-\frac{1}{3}y&=1, \\ 6(x+y)-3(x-y)&=39, \end{aligned}\right\}$

(85) $\left.\begin{aligned} \frac{2x}{3}-3y&=1\tfrac{2}{3}, \\ \frac{4x}{15}-y&=1\tfrac{2}{3}, \end{aligned}\right\}$

(86) $\left.\begin{aligned} \frac{3}{x}+\frac{4}{y}&=2, \\ \frac{4}{x}+\frac{3}{y}&=2\tfrac{1}{12}, \end{aligned}\right\}$

(87) $\left.\begin{aligned} \frac{4y-6}{x+y}&=2, \\ \frac{8x-5}{y-x}&=9, \end{aligned}\right\}$

(88) $\left.\begin{aligned} \frac{5}{3}x-y&=\frac{1}{2}, \\ 8x+\frac{3}{4}y&=4\tfrac{1}{4}, \end{aligned}\right\}$

(89) $\left.\begin{aligned} \frac{3}{4}y+\frac{1}{2}x&=17, \\ \frac{1}{2}y-\frac{1}{3}x&=4\tfrac{2}{3}. \end{aligned}\right\}$

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(90) Two-thirds of a certain number of persons received 1*s.* 6*d.* each, and one-third received 2*s.* 6*d.* each; the whole sum distributed was £2. 15*s.*, what was the number of persons?

(91) I have $3\frac{1}{2}$ times as many shillings as half-crowns, and altogether my money amounts to 4 guineas. How many of each coin have I?

(92) Divide 17 into two such parts that one of them shall contain the other 17 times exactly.

(93) Divide the decimal fraction 0·03 into two others, which differ from each other by 0·003.

(94) *A* travels at a certain rate: had he gone half a mile per hour faster, he would have done the journey in four-fifths of the time; whereas had he gone half a mile per hour slower, he would have been $2\frac{1}{2}$ hours longer on the road. What distance did he go? And at what speed?

(95) At what time after 3 o'clock are the hour and minute hands of a watch together?

(96) A wine-merchant has two sorts of wine, one of which he sells at 3*s.*, and the other at 1*s.* 8*d.* per quart.

He wishes to have a mixture of the two amounting to 50 quarts, which he may sell at 2*s.* 6*d.* per quart, and make the same profit. How much of each sort must he take?

(97) The rent of a farm is paid in wheat and barley. When wheat is at 55*s.* per quarter, and barley at 33*s.*, the portions of rent for wheat and barley are equal; but when wheat is at 65*s.* and barley at 41*s.* the whole rent is increased by £7. What is the corn-rent?

(98) A certain number of sovereigns, shillings, and sixpences together amount to £8. 6*s.* 6*d.* in value; and the value of the shillings is less by a guinea than that of the sovereigns, and greater by a guinea and a half than that of the sixpences. How many of each coin were there?

(99) A sum of money is divided in equal portions among a number of persons: if there had been 5 persons more, each would have received 1*s.* 9*d.* less: and if there had been 3 fewer, each would have received 1*s.* 9*d.* more. What was the sum divided? And what was the number of persons?

(100) Find the number which increased by 2 is contained exactly as many times in 30, as the same number diminished by 1 is contained in 15.

MISCELLANEOUS EXERCISES.

SECOND SERIES.

[The following Examples and Problems are *for the most part* not original, but selected from College Examination Papers and various other quarters.]

Simplify the following algebraical expressions:—

(1) $(2c-3r)x-(c-1)x-(c-2r)x-x$.

(2) $(q-b)x^2-(q+b)x^2+3bx^2-2x^2$.

(3) $(a-2p)x^3+(a+2p)x^3-(p-a)x^3-x^3$.

(4) Is it correct to make $\dfrac{1}{a-b}$ equal to $\dfrac{1}{a}-\dfrac{1}{b}$?

(5) Is it correct to make $\frac{a-b}{x}$ equal to $\frac{a}{x}-\frac{b}{x}$?

(6) What is the value of $\frac{ab^2-2abc+c^2}{b^2-3bc+2c^2}$, when $a=b=-c$

(7) From $2(a+b)-3(c-d)$ subtract $a+b-4(c-d)$

(8) From $(a+b)x+(b+c)y$ subtract $(a-b)x-(b-c)y$

(9) From $6x-\frac{2a}{b}$ subtract $5\frac{1}{2}.x-\frac{a}{b}$.

(10) From $\frac{x+5}{4(x-1)}$ subtract $\frac{5x-15}{4(x-1)}$.

(11) Add together $\frac{n}{n+1}$ and $\frac{n^2}{n+1}$.

(12) Multiply $\frac{x}{2}+\frac{x}{3}$ by 6.

(13) Divide $1+x$ by $\frac{1}{x}+1$.

(14) Divide a^4+4b^4 by $a^2-2ab+2b^2$.

(15) Divide $7x^3+x-5x^2-3x^4$ by $1-3x$.

(16) Divide $a+b+\frac{a^2}{b}$ by $a+b+\frac{b^2}{a}$.

(17) Divide $a-\frac{1}{2}(a-\frac{2}{3}b)$ by $b-\frac{1}{3}(a+\frac{3}{2}b)$.

(18) Multiply $x+1+\frac{1}{x}$ by $x-1+\frac{1}{x}$.

(19) Divide $a^4-\frac{1}{a^4}$ by $a-\frac{1}{a}$.

(20) Square $\frac{1}{2}x-\frac{1}{x}$.

(21) Find the product of
$(ax+a^2+x^2)(x-a)(x^2-ax+a^2)(a+x)$.

(22) Divide $a-b$ by $\sqrt{a}-\sqrt{b}$.

(23) Add together $\frac{1}{2}.\frac{2x+3y}{2x-3y}$ and $\frac{1}{2}.\frac{2x-3y}{2x+3y}$.

(24) Simplify $\frac{x(x+1)(x+2)}{3}-\frac{x(x+1)(2x+1)}{1\times2\times3}$.

Find the values of the unknown quantities in the following equations :—

(25) $\frac{30}{x+2}=\frac{15}{x-1}$.

(26) $\frac{128}{3x-4}=\frac{216}{5x-6}$.

(27) $\frac{42x}{x-2}=\frac{35x}{x-3}$.

(28) $\frac{x^2-9}{3}=\frac{x+3}{4}$.

(29) $\frac{3}{x+1}=8-2\left(\frac{4x+3}{x+3}\right)$.

(30) $\frac{3}{1-3x}-\frac{4}{1-2x}=\frac{5}{5x-1}$.

(31) $\frac{1}{x+3}+\frac{2}{x+6}=\frac{3}{x+9}$.

(32) $\frac{1}{4}\{3x-\frac{1}{3}(x-1)\}=\frac{5}{6}x-1$.

(33) $\frac{x-3}{2\frac{1}{2}}-\frac{x-4}{6\frac{1}{3}}=\frac{14-x}{5}$.

(34) $\frac{2x-1}{2x+1}+\frac{2x+1}{2x-1}=3$.

(35) $\frac{48}{x+3}=\frac{165}{x+10}-5$.

(36) $3(x-\frac{1}{4})-\frac{x-1}{x+2}=5$.

(37) $\frac{1\frac{1}{2}}{5-x}+\frac{1}{4-x}=\frac{4}{2+x}$.

(38) $\frac{7x+1}{6\frac{1}{2}-3x}=\frac{80}{3}\left(\frac{x-\frac{1}{2}}{x-\frac{2}{3}}\right)$.

(39) $\frac{1}{2}(x-1)(x-2)=2\frac{1}{4}(x-2\frac{2}{3})$.

(40) $\frac{1}{2}(x+3)(2x-5)=6\frac{1}{5}(2x-6\frac{24}{31})$.

(41) $\frac{2x(a-x)}{3a-2x}=\frac{a}{4}$.

(42) $\frac{x^4+3x^3+6}{x^2+x-4}=x^2+2x+15$.

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(43) $\left.\begin{aligned}13x+135y&=374,\\ 123x+308y&=1600,\end{aligned}\right\}$

(44) $\left.\begin{aligned}11x+19y&=101,\\ 29x-37y&=5,\end{aligned}\right\}$

(45) $\left.\begin{aligned}56x+278&=47(x+y),\\ 28y+832&=17(x-y),\end{aligned}\right\}$

(46) $\left.\begin{aligned}2x+3y&=2\tfrac{1}{5}(x+4),\\ 3(x+y)&=5(x-y),\end{aligned}\right\}$

(47) $\left.\begin{aligned}5(\tfrac{1}{2}x-1)&=\tfrac{3}{2}(y+1)-\tfrac{1}{2},\\ \tfrac{1}{2}(y-5)&=3\tfrac{1}{3}(\tfrac{2}{5}-\tfrac{1}{10}x),\end{aligned}\right\}$

(48) $\left.\begin{aligned}\frac{x+2}{y+2}&=\frac{1}{2},\\ \frac{x-2}{y-2}&=\frac{1}{3},\end{aligned}\right\}$

(49) $\left.\begin{aligned}\frac{x+2}{y+6}&=\frac{1}{2},\\ \frac{x-2}{y+3}&=\frac{1}{3},\end{aligned}\right\}$

(50) $\left.\begin{aligned}\frac{x+7}{y}&=\frac{3}{2},\\ \frac{x}{y+10}&=\frac{1}{2},\end{aligned}\right\}$

(51) $\left.\begin{aligned}\frac{x}{y}+x+xy&=13,\\ x^2&=9.\end{aligned}\right\}$

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(52) A pile is one-fifth of its whole length in the earth, three-sevenths of its length in the water, and 13 feet out of the water, what is the length of the pile?

(53) One-third of a ship belongs to *A*, and one-fifth to *B*, and *A*'s part is worth £1000. more than *B*'s. What is the value of the ship?

(54) In a company of 90 persons, men, women, and children, there are 4 more men than women, and 10 more children than men and women put together. How many are there of each?

(55) A person is now 40 years old, and his son 9 years. In how many years will the father, who is now more than 4 times as old, be only twice as old, as his son?

(56) Two carpenters *A* and *B* received £5. 17*s*. for work done, *A* having worked 15, and *B* 14, days; and *A*'s wages for 4 days exceeded *B*'s for 3 days by 11*s*. What did each receive per day?

(57) Seven horses and four cows consume a stack of hay in 10 days, and two horses can eat it alone in 40 days; in how many days will one cow be able to eat it?

(58) A person was asked to state the ages of himself, of his father, and of his grandfather. He replied, 'My age and my father's amount together to 56 years, mine and my grandfather's to 80 years, and my father's and grandfather's to 100 years. What was the age of each?

(59) A boy spends 5*s*. in apples and oranges, buying the former at 6 a penny, and the latter at 4 a penny. He afterwards sold two-thirds of his apples and half his oranges for 3*s*. taking only cost price. How many of each fruit did he buy?

(60) A wine merchant has 4 dozen bottles of wine of a superior quality, which he must sell at 4 guineas a dozen, to make the necessary profit. But to get rid of it the sooner, he mixes with it just so much of an inferior wine worth 24*s*. per dozen as will enable him honestly to sell the mixed wine at 54*s*. per dozen, and obtain the same profit upon the superior wine. How much of the inferior wine is used?

(61) Divide the number n into two parts, so that one shall be n times as great as the other.

(62) A person was about to relieve a certain number of poor persons by giving them 2*s.* 6*d.* each, but found he had not money enough in his pocket by 3*s.* He then gave them 2*s.* each, and had 4*s.* to spare at last. How much money had he, and how many persons did he relieve?

(63) A hare is started at a distance equal to 50 of its own leaps before a greyhound, and takes 4 leaps to the greyhound's 3; but 2 of the greyhound's leaps cover as much ground as 3 of the hare's. How many leaps will the greyhound take to catch the hare?

(64) A father leaves to his children a certain sum which is to be divided as follows:—The oldest is to receive £100, and the 10th part of the remainder: the second is to have £200, and the 10th part of what then remains: the third £300, and the 10th part of the remainder; and so on, to the last. Now it is found that all the children have, by this complicated scheme, received exactly the same sum. What was the whole fortune, and the number of children?

(65) A certain waggon has a mechanical contrivance which marks the difference of the number of revolutions of the fore and hind wheels in any journey. The rim of each fore-wheel is $5\frac{1}{4}$ feet, and of each hind-wheel $7\frac{1}{8}$ feet: find the distance travelled when the fore-wheel has made exactly 2000 revolutions more than the hind-wheel.

(66) A person has a certain number of sovereigns which he tries to arrange in the form of a square, placing them as close together as possible on the table. At the first trial he had 130 sovereigns over; but when he enlarged the side of the square by 3 sovereigns, he had only 31 over. How many sovereigns had he?

(67) A farmer bought a certain number of sheep for £94: He lost 7 of them, and sold one-fourth of the remainder at *prime cost* for £20. How many did he buy?

(68) In three drawers there is altogether a sum of £162. In order that each drawer may contain the same, I take out of the first to put into each of the other two the half of what each already contained. I then take out of the second, and afterwards out of the third, and each time put into the two other drawers half of what each already contained. I have thus attained my object. How much was in *each* drawer at first?

(69) A person bought a quantity of cloth for £12. If he had bought one yard less for the same money it would have cost 1*s*. a yard more. How many yards did he buy?

(70) Divide 100 into two parts, so that the difference of their squares shall be 400.

(71) The sum of two fractions is $\frac{62}{63}$, their difference $\frac{8}{63}$, and both fractions are in lowest terms. What are the fractions?

(72) A traveller set out from a certain place, and went 1 mile the 1st day, 3 the 2nd, 5 the next, and so on, increasing by 2 miles every day. After he had been gone 3 days, another sets out from the same place on the same road, and goes 12 miles the first day, 13 the 2nd, and so on. In how many days will the latter overtake the former?

(73) The numbers of boys in the 3 classes of a school were as the numbers, 5, 7, 8. At the next inspection the 1st class was found increased by 4 boys, the 2nd had gained two-sevenths of its former number, the 3rd was doubled,—and the whole number of additional scholars was 34. What were the numbers in the classes at the first inspection?

(74) If the interest of the National Debt be reckoned 30 millions sterling per annum, and 3 per cent. the average rate of interest paid, what reduction in the rate of interest would give the same relief to taxation as the paying off 200 millions of debt, and allowing the interest paid on the remainder to continue the same?

(75) The *Specific Gravity* of silver is $10\frac{1}{2}$, that of copper 9, and a certain compound of the two is found to be of *Specific Gravity* $10\frac{1}{11}$. What quantity of each metal is there in 148 lbs. of the mixture?

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(76) If $a : b :: b : c$, and $b : c :: c : d$, shew that $a : d :: a^3 : b^3$; and that $a+b : b+c :: b+c : c+d$.

(77) If $6x-a : 4x-b :: 3x+b : 2x+a$, find x.

(78) If $a : b :: c : d$, shew that

$$a : a+b :: a+c : a+b+c+d.$$

(79) Divide 20 into 3 parts, such that the ratio of the first two shall be 2 : 5, and that of the last two 5 : 3.

(80) Find two numbers in the ratio $1\frac{1}{2} : 2\frac{2}{3}$, and such, that when each number is increased by 15, they shall be in the ratio $1\frac{2}{3} : 2\frac{1}{2}$.

(81) A quantity of milk is increased by water in the ratio of 4 : 5, and then 3 gallons are sold; the rest being mixed with 3 quarts of water is increased in the ratio of 6 : 7. How many gallons of milk were there at first?

(82) Given that the solid content of a globe *varies* as the cube of its diameter, what ratio does the content of a globe whose diameter is 4 inches bear to that of one whose diameter is 8 inches?

(83) Given that the illumination from a source of light *varies inversely* as the *square* of the distance, how much farther from a candle must a book, which is now 8 inches off, be removed, so as to get *half* as much light?

(84) Given that the content of a cylinder *varies* as its height and the *square* of its diameter *jointly*, compare the contents of two cylinders, one of which is *twice* as high as the other, but with only *half* its diameter.

(85) If a servant agree with his master to receive, for his wages, a farthing for the 1st month, a penny for the 2nd, four pence for the 3rd, and so on; what will 12 months' wages amount to?

(86) Distribute 250 policemen among 4 towns in *proportion* to their respective populations, which are 5300, 2940, 680, and 1870.

(87) Two globes of metal whose diameters are 6 in. and 7 in., are melted down and together formed into a single globe; what is the diameter of the new globe? (See 82.)

(88) On the 1st of Jan. 1799, a poor man received from *A* as many groats as *A* was years old, and a similar gift each January for the seven years following, in the last of which *A* died, his alms to the poor man having amounted in all to £7. 18*s*. 8*d*. What was *A*'s age when he died? And in what year was he born?

ANSWERS TO THE EXERCISES.

EXERCISES. A.

(1) 20.
(2) 6.
(3) 14.
(4) 0.
(5) 13.
(6) 35.
(7) 62.
(8) 10.
(9) 39.
(10) 62.
(11) 0.
(12) 2.

(13) $3a$.
(14) $6ab$.
(15) $6a$.
(16) 2, $2b$, bx, $3bx$, m, xx, px, bxy.
(17) 5.
(18) 11.
(19) 12.
(20) 7.

(21) 30.
(22) 10.
(23) 9.
(24) 3.
(25) 1.
(26) 23.
(27) 14.
(28) $1\frac{1}{2}$.
(29) 3.
(30) $m+n-p$.

EXERCISES. B.

(1) 18.
(2) 0.
(3) 114.
(4) 657.
(5) 0.
(6) $49\frac{1}{9}$.
(7) 23.
(8) $m+81n-64p$.
(9) 2.
(10) 6.
(11) 25.
(12) 1.
(13) $3m+6n-4p$.
(14) 4.
(15) 2.
(16) 1.

(17) 2.
(18) 8.
(19) 20.
(20) 6.
(21) 1.

EXERCISES. C.

(1) $2a + 2b$.
(2) $2a$.
(3) $2a - 2b$.
(4) $2a$.
(5) $2a + 2c$.
(6) $2 + m + n$.
(7) $7m - 1$.
(8) $4xy + 4x$.
(9) $p - q + 8$.
(10) $6ab - bc$.
(11) $3ax + 2by$.
(12) $5a - 5b + 5c$.
(13) $4xy - x - 4$.
(14) $3q - 2p + pq$.
(15) $2p^2 + 2q^2$.
(16) $8ab + ac - 1$.
(17) $4x + 3y$.
(18) $2 + 5a$.
(19) $2a + 6c$.
(20) $3a^2 + ab - 2b^2$.
(21) $6 - 5x$.
(22) $2ac + 2bd$.
(23) $2ax - 2by$.
(24) $5x^2y - 3axy - a^2x + x^3$.
(25) $mn + m - n + 1$.
(26) $3z - 2y$.
(27) $2x^3 + 2a^3$.
(28) $a^2 + b^2 + c^2$.
(29) $x^2 + xy + y^2 + mx + ny$.
(30) $\frac{3}{2}ad + \frac{3}{2}bd - cd + \frac{1}{2}ab - ac$.

EXERCISES. D.

(1) $a - b + x$.
(2) $2b - 2c$.
(3) $5a - 3c$.
(4) $8a - 7b$.
(5) $x - y - 9z$.
(6) $ax + 2by - 2c$.
(7) $bc - 2ab + 2a$.
(8) $2x^2$.
(9) $xy + 5x^2 + 5y^2$.
(10) $mn + 4m - 4n$.
(11) $xy + 3mx$.
(12) $3abc - 3ab - 2ac - 1$.
(13) $b^2 + 3c^2$.
(14) $2ax - 2a^2 - 2x^2$.
(15) $2a^2b + 3a^2c + 2c^3$.
(16) $2xy + a - 1$.
(17) $\frac{1}{3}ax - xy + 1$.
(18) $\frac{1}{2}a + \frac{3}{2}b - \frac{1}{2}c$.
(19) 45, and 15, years.
(20) $\frac{3}{4}$, and $\frac{1}{4}$.

EXERCISES. E.

(1) $abxy$.
(2) $-3mnp$.
(3) $-8axy$.
(4) $8axy$.
(5) abc.
(6) $3m^2np$.
(7) $3m + 3n - 3p$.
(8) $apx + bpx^2$.
(9) $2a^2d + 4abd$.

(10) $4a^2x - 2a^2x^2y$.
(11) $-3x^2y + 2x^2y^2 - 6xy$.
(12) $-3n + 6nax - 9nbx^2$.
(13) $-4abx + 6acx - 10bdx$.
(14) $14x^2y - 21x$.
(15) $4ax^2yz + 2bxy^2z - 2cxyz^2$.
(16) $2a^2by - b^2xy + bdy$.

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(17) $ab + bx + ay + xy$.
(18) $6x^2 - 2x - 4$.
(19) $x^2 - x - 12$.
(20) $6x^2 - 19x + 10$.
(21) $1 - x^2$.
(22) $x - 3x^2 + 2x^3$.
(23) $2ax^2 + 2bxy - axy - by^2$.
(24) $a^2 - ax - 6x^2$.
(25) $35x^2 - 33x + 4$.
(26) $8axy - 12by^2 - 6ax^2 + 9bxy$.
(27) $2m + n - 4m^2n - 2mn^2$.
(28) $a^3c - abc^2 - a^2b^2 + b^3c$.
(29) $x - y + 2x^2 + xy - 3y^2$.
(30) $ab + bx - by - ay - xy + y^2$.
(31) $2a^2c - 3abc + b^2c + 2a^2d - abd$.

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(32) $a^4 - 1$.
(33) $x^4 - a^4$.
(34) $8x^3 + 27$.
(35) $16 + 4x^2 + x^4$.
(36) $a^6 - 3a^2x^2 + 2x^4$.
(37) $x^4 - 81$.
(38) $4a^6x^4 - 9b^4y^2$.
(39) $4a^4 - 9a^2b^2 + 6ab^3 - b^4$.
(40) $a^{10} - 1$.

EXERCISES. F.

(1) x.
(2) 7.
(3) $7x$.
(4) a.
(5) $3x$.
(6) a.
(7) $-ay$.
(8) $-ay$.
(9) $-3a$.
(10) $2axy$.
(11) $-14nx$.
(12) $2bx$.

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(13) $3c - 2bd$.
(14) $2c - bd$.
(15) $-4x + 3y$.
(16) $1 + 8ac - 2bc$.
(17) $-2ax + 4b + 1$.
(18) $a^2 - 5bx + 6x^2$.

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(19) $x + 1$.
(20) $c + d$.
(21) $3 - b$.
(22) $2a - 5x$.
(23) $a + 2$.
(24) $2ab$.
(25) $3x - 5$.
(26) $3x^2 - x + 2$.
(27) $a - b - c$.
(28) $5a^2 + 3x^2$.
(29) $p^2q + 4pq^2 + 2q^3$.
(30) $ax^2 - bx^2 - a^2x + abx + a^3 - a^2b$.
(31) $16x^4 - 24x^3 + 36x^2 - 54x + 81$.

EXERCISES. G.

(1) 4.
(2) 25.
(3) 20.
(4) x.
(5) bx^2.
(6) apx.
(7) $5abx$.
(8) $3a^2b^2$.
(9) $9a^4b^2c^6$.
(10) $7mnp$.
(11) axy.
(12) $\frac{2}{5}a$.
(13) d.
(14) x.

EXERCISES. H.

(1) 168.
(2) 240.
(3) 56.
(4) 168.
(5) 2520.
(6) 42504.
(7) abx.
(8) $2axy$.
(9) $24x$.
(10) abc.
(11) $2x^2y^2$.
(12) bc^2d^2.

EXERCISES. I.

(1) $\frac{2a}{3}$.
(2) $2b$.
(3) $\frac{4bx}{3a}$.
(4) $\frac{bx}{2}$.
(5) $\frac{5x^2}{a}$.
(6) $\frac{1}{2bx}$.
(7) $\frac{m-n}{mn}$.
(8) $\frac{2x-3}{5}$.
(9) $\frac{2a+3}{b}$.
(10) $\frac{2b+1}{a}$.
(11) $\frac{3a-2x}{2a-3x}$.
(12) $\frac{n-m+p}{m-n+p}$.

EXERCISES. J.

(1) $\frac{6x}{5}$.
(2) $\frac{5ab}{6}$.
(3) $\frac{3a+1}{3}$.
(4) $\frac{2a}{5}$.
(5) $\frac{6x-4}{7}$.
(6) $\frac{10x-2}{21}$.
(7) $\frac{6}{a}$.
(8) $\frac{6}{ab}$.
(9) $\frac{x^2+y^2+2xy}{x^2y^2}$.
(10) $\frac{bc+2c+3}{abc}$.
(11) $\frac{19x-23}{6}$.
(12) $\frac{34x-23}{12}$.
(13) $\frac{54x-13}{50}$.
(14) $\frac{62}{15x}$.
(15) $\frac{41}{20y}$.
(16) $\frac{bcx+acy+abz}{abc}$.
(17) $\frac{axy+cxy}{abc}$.
(18) 0.

(19) $\frac{x}{10}$.

(20) $\frac{x}{8}$.

(21) $\frac{1}{2}$.

(22) $\frac{x+5}{8}$.

(23) $\frac{5x-y-3}{10}$.

(24) $\frac{4x+16}{x+1}$.

(25) $\frac{2}{x}$.

(26) $\frac{2x^2+x}{x^2+3x+2}$.

(27) $\frac{5x+35}{42}$.

(28) $\frac{17x-34}{50}$.

(29) $\frac{acx}{b^2+bcx}$.

(30) $\frac{x^2+y^2}{xy+y^2}$.

(31) $\frac{1}{1+x^2+2x}$.

(32) $\frac{4xy}{x^2-y^2}$.

EXERCISES. K.

(1) $\frac{3x}{2}$.

(2) $3x$.

(3) $\frac{5x}{2}$.

(4) $2x$.

(5) $2a-2x$.

(6) $28x$.

(7) $8x$.

(8) $9x-15$.

(9) $60+45x$.

(10) $16-14x$.

(11) $72x+156$.

(12) $4x-2$.

(13) $6x+8$.

(14) $3x-5$.

(15) $10-x$.

(16) $\frac{3x}{4}$.

(17) x^2.

(18) $\frac{2-3x}{5}$.

(19) $\frac{1}{x^2}$.

(20) $x^2+y^2-\frac{1}{x^2}-\frac{1}{y^2}$.

(21) $\frac{x}{2}$.

(22) $\frac{3x}{20}$.

(23) $\frac{x}{8}$.

(24) $\frac{3x}{4y}$.

(25) $\frac{m}{p}$.

(26) $\frac{1-2y}{y}$.

(27) $\frac{1+2b}{4}$.

(28) $\frac{5y^2}{2z}$.

(29) $-\frac{2b^2}{3}$.

(30) $\frac{2ax^2}{bc}$.

(31) $x^2+2+\frac{1}{x^2}$.

(32) $y+\frac{1}{y}+2$.

(33) $\frac{1}{1-x^2}$.

(34) 1.

(35) $\frac{m^2-3m+2}{6}$.

(36) $\frac{3}{4}+\frac{1}{2}\cdot\frac{b^2}{a^2}-\frac{1}{2}\cdot\frac{a^2}{b^2}$.

(37) ab.

(38) $\frac{a^2-x^2}{a^2+x^2}$.

(39) $\dfrac{2x+1}{x-2}$.

(40) $\dfrac{2-3x+x^2}{xy}$.

(41) $\dfrac{2b-6a}{2ab-b^2}$.

(42) $\dfrac{4-x}{4}$.

(43) $\dfrac{1}{1-x}$.

(44) $\dfrac{1}{x}$.

(45) $\dfrac{a^2-ax}{b}$.

(46) $\dfrac{a^2+ax+x^2}{a^2-ax+x^2}$.

EXERCISES. L.

(1) ac.
(2) $4-x$.
(3) $4x$.
(4) $2a^2-2b^2$.
(5) $7+5x$.
(6) $\dfrac{5a}{2}-\dfrac{3x}{2}$.
(7) b.
(8) ax^2+bx^2.
(9) $6-5x$.
(10) $1-x$.
(11) $5a-3c$.
(12) a^2.
(13) $1-x^4$.
(14) x.
(15) $\dfrac{a}{b}$.
(16) $\dfrac{b}{a}$.
(17) $\dfrac{2+x}{1-x^2}$.
(18) 1.
(19) x^2+x.
(20) $4x^2-x^4$.

EXERCISES. M.

(1) $x=6$.
(2) $x=1$.
(3) $x=6$.
(4) $x=8$.
(5) $x=3$.
(6) $x=4$.
(7) $x=2$.
(8) $x=1$.
(9) $x=10$.
(10) $x=8$.
(11) $x=5$.
(12) $x=12$.
(13) $x=1$.
(14) $x=\frac{1}{2}$.
(15) $x=4$.
(16) $x=12$.
(17) $x=9$.
(18) $x=7$.
(19) $x=10$.
(20) $x=30$.
(21) $x=5$.
(22) $x=5$.
(23) $x=7$.
(24) $x=7$.
(25) $x=7$.
(26) $x=14$.
(27) $x=60$.
(28) $x=84$.
(29) $x=35$.
(30) $x=5$.
(31) $x=7$.
(32) $x=2$.
(33) $x=9$.
(34) $x=7$.
(35) $x=4$.
(36) $x=8$.

EXERCISES. N.

(1) $x=5$.
(2) $x=5$.
(3) $x=\frac{2}{3}$.
(4) $x=10$.
(5) $x=6\frac{1}{2}$.
(6) $x=4\frac{7}{9}$.
(7) $x=3$.
(8) $x=7$.
(9) $x=14$.
(10) $x=8$.
(11) $x=7$.
(12) $x=2$.

EXERCISES. O.

(1) $x=4$.
(2) $x=\frac{1}{6}$.
(3) $x=\frac{1}{20}$.
(4) $x=\frac{1}{ab}$.
(5) $x=18$.
(6) $x=8$.
(7) $x=8$.
(8) $x=8$.
(9) $x=12$.
(10) $x=\frac{1}{4}$.
(11) $x=2$.

EXERCISES. P.

(1) 16.
(2) 12.
(3) 18.
(4) 60.
(5) 10.
(6) 10, and 30.
(7) $10\frac{5}{7}$, and $14\frac{2}{7}$.
(8) $1\frac{1}{6}$, and $8\frac{1}{6}$.
(9) 6*s*. 6*d*., 5*s*. 6*d*., 4*s*. 6*d*., and 3*s*. 6*d*.
(10) 5, $6\frac{1}{2}$, 9, and $12\frac{1}{2}$, feet.
(11) 8, and 16.
(12) 8.
(13) 8, and 40.
(14) 24, and 6, yrs.
(15) 35, 36, and 71.
(16) 44, and 36.
(17) $\frac{4}{5}$.
(18) $\frac{3}{5}$.
(19) 240.
(20) $27\frac{3}{11}$m. before 1.
(21) $27\frac{3}{11}$m. past 5.
(22) 2 miles.
(23) $4\frac{2}{7}$ miles.
(24) 22, 7, 12, gall.
(25) 3*s*.$4\frac{8}{25}$*d*., 1*s*.$8\frac{4}{25}$*d*.

EXERCISES. Q.

(1) $\left.\begin{array}{l}x=12,\\ y=5.\end{array}\right\}$
(2) $\left.\begin{array}{l}x=10,\\ y=2.\end{array}\right\}$
(3) $\left.\begin{array}{l}x=6,\\ y=2.\end{array}\right\}$
(4) $\left.\begin{array}{l}x=3,\\ y=1.\end{array}\right\}$
(5) $\left.\begin{array}{l}x=1,\\ y=2.\end{array}\right\}$
(6) $\left.\begin{array}{l}x=7,\\ y=10.\end{array}\right\}$
(7) $\left.\begin{array}{l}x=4,\\ y=3.\end{array}\right\}$
(8) $\left.\begin{array}{l}x=2,\\ y=3.\end{array}\right\}$
(9) $\left.\begin{array}{l}x=2,\\ y=3.\end{array}\right\}$
(10) $\left.\begin{array}{l}x=11,\\ y=7.\end{array}\right\}$
(11) $\left.\begin{array}{l}x=\frac{1}{4},\\ y=\frac{1}{5}.\end{array}\right\}$

(12) $\left.\begin{array}{l} x=5, \\ y=4. \end{array}\right\}$

(13) $\left.\begin{array}{l} x=10, \\ y=7. \end{array}\right\}$

(14) $\left.\begin{array}{l} x=5, \\ y=3. \end{array}\right\}$

(15) $\left.\begin{array}{l} x=6, \\ y=10. \end{array}\right\}$

(16) $\left.\begin{array}{l} x=3, \\ y=10. \end{array}\right\}$

(17) $\left.\begin{array}{l} x=3, \\ y=\frac{1}{2}. \end{array}\right\}$

(18) $\left.\begin{array}{l} x=6, \\ y=2. \end{array}\right\}$

(19) $\left.\begin{array}{l} x=8, \\ y=9. \end{array}\right\}$

(20) $\left.\begin{array}{l} x=8, \\ y=8. \end{array}\right\}$

EXERCISES. R.

(1) $\left.\begin{array}{l} x=2, \\ y=1. \end{array}\right\}$

(2) $\left.\begin{array}{l} x=11, \\ y=9. \end{array}\right\}$

(3) $\left.\begin{array}{l} x=6, \\ y=4. \end{array}\right\}$

(4) $\left.\begin{array}{l} x=8, \\ y=\frac{1}{2}. \end{array}\right\}$

(5) $\left.\begin{array}{l} x=4, \\ y=21. \end{array}\right\}$

(6) $\left.\begin{array}{l} x=144, \\ y=216. \end{array}\right\}$

(7) $\left.\begin{array}{l} x=114, \\ y=77. \end{array}\right\}$

(8) $\left.\begin{array}{l} x=40, \\ y=16. \end{array}\right\}$

(9) $\left.\begin{array}{l} x=5, \\ y=9. \end{array}\right\}$

(10) $\left.\begin{array}{l} x=13, \\ y=3. \end{array}\right\}$

(11) $\left.\begin{array}{l} x=7, \\ y=10. \end{array}\right\}$

(12) $\left.\begin{array}{l} x=7, \\ y=4. \end{array}\right\}$

EXERCISES. S.

(1) 22, and 26.

(2) £15, and £35.

(3) 24 men, 20 women.

(4) 15 men, 22 women.

(5) $\frac{7}{13}$.

(6) $\frac{1}{5}$.

(7) 14, and 6.

(8) 12, and 18.

(9) 11, and 5, gall.

(10) A.D. 1752.

EXERCISES. T.

(1) $25a^2x^2$.

(2) $25a^2x^2y^2$.

(3) $49a^2b^2$.

(4) $a^4b^2c^2$.

(5) $49a^4b^2c^6$.

(6) $\frac{a^2b^2}{c^2}$.

(7) $\frac{9a^2x^2}{4b^2y^2}$.

(8) $\frac{a^4b^2}{4c^2}$.

(9) $\frac{16a^4b^2}{49x^6y^6}$.

(10) $\frac{9x^2y^4}{4x^4}$.

(11) $\frac{16}{25a^4b^2c^6}$.

(12) a^2+1+2a.

(13) $a^2b^2+1+2ab$.

(14) x^2+9+6x.

(15) $4+y^2-4y$.

(16) $4m^2+n^2-4mn$.

(17) $4x^2+9y^2-12xy$.

(18) $x^2+\frac{p^2}{4}-px$.

(19) $x^2+\frac{9}{4}+3x$.

(20) $m^2x^2+n^2+2mnx$.

(21) $4m^2x^2+n^2-4mnx$.

(22) $a^2b^2x^2+c^2+2abcx$.

(23) $9x^2y^2+a^2-6axy$.

(24) $\frac{1}{4}a^2b^2+c^2+abc$.

EXERCISES. U.

(1) $2ab$.

(2) $3xy^2$.

(3) $10ab^2c^2$.

(4) $\frac{3ax}{2b}$.

(5) $\frac{2ab}{3xy^2}$.

(6) $\frac{1}{2}\cdot\frac{mx^2}{ny}$.

(7) $1-x$.

(8) $2x+1$.

(9) $2a-b$.

(10) $3x+1$.

(11) $x+\frac{1}{2}$.

(12) $x-\frac{1}{x}$.

(13) $x^2-12x+36$.

(14) $x^2-14x+49$.

(15) $x^2+11x+\frac{121}{4}$.

(16) x^2+2x+1.

(17) $x^2-x+\frac{1}{4}$.

(18) $x^2+\frac{4x}{5}+\frac{4}{25}$.

(19) $x^2-\frac{2x}{7}+\frac{1}{49}$.

(20) $x^2+\frac{1}{2}x+\frac{1}{16}$.

(21) $x^2-\frac{1}{3}x+\frac{1}{36}$.

(22) $x^2-\frac{5}{6}x+\frac{25}{144}$.

(23) $x^2-\frac{3x}{4}+\frac{9}{64}$.

(24) $x^2-\frac{7x}{10}+\frac{49}{400}$.

EXERCISES. V.

(1) $x=\pm 6$.

(2) $x=\pm 4$.

(3) $x=\pm 1$.

(4) $x=\pm 4$.

(5) $x=\pm 2$.

(6) $x=\pm 5$.

(7) $x=\pm 5$.

(8) $x=\pm 3$.

(9) $x=\pm\frac{1}{2}$.

(10) $x=\pm 2$.

(11) $x=\pm 2$.

(12) $x=1\frac{1}{4}$, or $\frac{1}{4}$

EXERCISES. W.

(1) $x=5$, or -2.
(2) $x=4$, or 1.
(3) $x=8$, or 2.
(4) $x=20$, or -6.
(5) $x=2$, or 10.
(6) $x=2$.
(7) $x=6$, or 1.
(8) $x=6$, or -5.
(9) $x=1\frac{1}{2}$, or -2.
(10) $x=6$, or $-4\frac{1}{2}$.
(11) $x=6$, or $-10\frac{1}{2}$.
(12) $x=1\frac{1}{2}$, or $\frac{3}{10}$.
(13) $x=\frac{2}{3}$, or -3.
(14) $x=6$, or $-10\frac{1}{2}$.
(15) $x=6$, or $-5\frac{2}{3}$.
(16) $x=1\frac{1}{2}$, or $-\frac{15}{22}$.
(17) $x=1$, or $\frac{2}{3}$.
(18) $x=2\frac{2}{3}$, or -2.
(19) $x=1\frac{1}{2}$, or $-\frac{5}{6}$.
(20) $x=2$, or $-1\frac{1}{9}$.
(21) $x=2$, or $-1\frac{2}{5}$.
(22) $x=4$, or -2.
(23) $x=7$, or $-\frac{1}{3}$.
(24) $x=1\frac{2}{3}$, or $-1\frac{1}{2}$.
(25) $x=\frac{1}{2}$, or $-1\frac{1}{3}$.
(26) $x=2$, or -3.
(27) $x=2$, or $\frac{1}{16}$.
(28) $x=2$, or $-\frac{1}{3}$.
(29) $x=16$, or -20.
(30) $x=11$, or -13.
(31) $x=3$, or $-\frac{4}{5}$.
(32) $x=4$, or $-1\frac{2}{3}$.
(33) $x=7$, or $-1\frac{3}{7}$.
(34) $x=3$, or $1\frac{2}{11}$.
(35) $x=8$, or $-\frac{14}{43}$.
(36) $x=9$, or $-\frac{25}{81}$.
(37) $x=1$, or $-1\frac{1}{9}$.
(38) $x=2$, or $4\frac{6}{13}$.
(39) $x=8$, or $13\frac{22}{31}$.
(40) $x=16$, or $-1\frac{1}{3}$.

EXERCISES. X.

(1) $\left.\begin{array}{l} x=\pm 4, \\ y=\pm 2. \end{array}\right\}$

(2) $\left.\begin{array}{l} x=\pm 8, \\ y=\pm 10. \end{array}\right\}$

(3) $\left.\begin{array}{l} x=\pm 12, \\ y=\mp 3. \end{array}\right\}$

(4) $\left.\begin{array}{ll} x=8, & \text{or } -3\frac{1}{2}, \\ y=3\frac{1}{2}, & \text{or } -8. \end{array}\right\}$

(5) $\left.\begin{array}{l} x=\frac{1}{2}, \text{ or } \frac{1}{4}, \\ y=\frac{1}{4}, \text{ or } \frac{1}{2}. \end{array}\right\}$

(6) $\left.\begin{array}{l} x=1, \text{ or } -12, \\ y=3, \text{ or } 11\frac{2}{3}. \end{array}\right\}$

(7) $\left.\begin{array}{l} x=2, \text{ or } -\frac{1}{3}, \\ y=4, \text{ or } 1\frac{2}{3}. \end{array}\right\}$

(8) $\left.\begin{array}{l} x=5, \text{ or } -9\frac{1}{4}, \\ y=3, \text{ or } -5\frac{5}{6}. \end{array}\right\}$

(9) $\left.\begin{array}{l} x=2, \text{ or } -1\frac{5}{13}, \\ y=3, \text{ or } -5\frac{6}{13}. \end{array}\right\}$

(10) $\left.\begin{array}{l} x=3, \text{ or } 2, \\ y=2, \text{ or } 3. \end{array}\right\}$

(11) $\left.\begin{array}{l} x=2, \text{ or } 5\frac{1}{5}, \\ y=4, \text{ or } -\frac{4}{5}. \end{array}\right\}$

(12) $\left.\begin{array}{l} x=7, \text{ or } -\frac{7}{43}, \\ y=6, \text{ or } -\frac{6}{43}. \end{array}\right\}$

EXERCISES. Y.

(1) 12, and 13.
(2) 3, 4, 5.
(3) 4, and 16.
(4) 14, and 196.
(5) 12, and 13.
(6) 20, and 10.
(7) 8, and 18.
(8) 13.
(9) $\frac{1}{2}$.
(10) 13, and 12, miles per hour.
(11) 9 miles per hour.
(12) 25, and 20.
(13) 54, and 48.
(14) 18, and 12, miles.
(15) 4, and 5, yards.
(16) 4 miles per hour forwards, and 1 mile backwards.

EXERCISES. Z.

(1) $\frac{1}{5}$.
(2) $\frac{1}{5x}$.
(3) $\frac{a}{b}$.
(4) $\frac{a}{1}$.
(5) $\frac{ay}{2}$.
(6) $\frac{3b}{2a}$.
(7) $\frac{ap}{3x}$.
(8) $\frac{x}{4y}$.
(9) $\frac{a+b}{c}$.
(10) $\frac{2a+x}{m}$.
(11) $\frac{1+x}{1}$.
(12) $\frac{a-b}{1}$.

(13) $5a : 4$.
(14) $4y : 5x$.
(15) $2a : 3b$.
(16) $8y : x$.
(17) $28x : 5y$.
(18) $(n-1)x : 2a$.

(19) $16 : 17$.
(20) $3a : 2b$.
(23) $a^2 - x^2 = ab$.
(24) $y^2 = 2ax - x^2$.
(25) $9 : 2$.
(26) 4, and 6.
(27) $2by$.
(28) 15, and 20.
(29) $y = 2ax$.
(30) $y = \frac{4}{x}$.

EXERCISES. *Za.*

(1) 71, and 96.
(2) 2, and -3.
(3) 5, and $6\frac{2}{3}$.

(4) 400.
(5) 670.
(6) 3900.
(7) 1430.
(8) -460.
(9) $57\frac{1}{2}$.
(10) $196\frac{2}{3}$.
(11) 78.
(12) £51. 3*s*. $9\frac{1}{4}$*d*. and 18*s*. $5\frac{1}{4}$*d*.
(13) £47. 10*s*.
(14) $19\frac{41}{66}$ miles.

(15) $\frac{13}{72}$.
(16) 1.
(17) $\frac{1}{4}(a + b)$.

(18) 8, 11.
(19) $1\frac{1}{2}$, 2, $2\frac{1}{2}$.
(20) 96, 92, 88, 84.
(21) 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$.
(23) $1\frac{40}{117}$, $1\frac{41}{117}$, &c. and 60.
(24) 100, and £767. 10*s*.
(25) $1\frac{1}{4}$ miles nearly.

EXERCISES. *Zb.*

(1) 2.
(2) 2.
(3) 3.
(4) $\frac{1}{2}$.
(5) $\frac{2}{3}$.
(6) 0·1, or $\frac{1}{10}$.
(7) 2.
(8) $2a$.
(9) $\frac{n}{r}$.

(10) 3, and 9.
(11) $\frac{1}{25}$, and $\frac{1}{125}$.
(12) $13\frac{4}{9}$.
(13) $4\frac{22}{27}$.

(14) 15.

(15) $\frac{1}{2}$.

(16) 20, 80.

(17) $\frac{1}{2}$, $\frac{1}{4}$.

(18) 18, 54, 162.

(19) 40, 16, $6\frac{2}{5}$.

(20) The former, by $\frac{2}{9}$.

(21) Yes. y.

(22) $\frac{1}{3}$, 1, 3, 9, &c

(23) 15 times.

MISCELLANEOUS EXERCISES.

FIRST SERIES.

(1) $2ab$, and xy.

(2) mx, and nx^2.

(3) a^2, b^2, and $2ab$.

(4) $2x$.

(5) 1.

(6) 6.

(7) Yes.

(8) $3m^2nx$.

(9) $2abxyz$.

(10) 50.

(11) 0.

(12) 28.

(13) 4, the inde of a.

(14) $\frac{12}{65}$.

(15) 6.

(16) $4a^2 - 9b^2$.

(17) $9m^2 - n^2$.

(18) $4b^4 - 9a^2b^2 - 24a^3b - 16a^4$.

(19) $105a^{12}$.

(20) $2a^{2n}$.

(21) $mna^{m+1}b^{n+1}$.

(24) x^2.

(25) $3a - 4x$.

(26) $4a - 2bc + 9xy$.

(27) $2a + 3$.

(28) $2b^2 - 3ab - 4a$

(29) $3a^n + b$.

(30) $2\times2\times2\times2\times3\times aabxxx$.

(31) $2\times2\times2\times2xxyyyy$,
$2\times2\times7axy$.
G.C.M. $4xy$.

(32) ab.

(33) $a + x$.

(34) $a + b$, and $a + b$.

(35) $a + b$, and $a - b$.

(36) $2x + a$, and $2x - a$.

(37) $4ab + 3x$, and $4ab - 3$

(38) $12a$.

(39) $60xy^2$.

(40) $24a^2x^3$.

(41) a.

(42) $\frac{1}{1+x}$.

(43) $\frac{1-x}{1+x}$.

(44) $\frac{2a+6x}{3}$.

(45) $\frac{ab}{m-p}$.

(46) $\frac{x+y}{x-y}$.

(47) $x+3$.

(48) $x+4$.

(49) $x+3$.

---

(50) $\frac{33a}{10b}$.

(51) $\frac{a+x}{a-x}$.

(52) $\frac{2mn}{n^2-x^2}$.

(53) $\frac{10}{x-2}$.

(54) $\frac{4a}{a^2-1}$.

(55) $a+1$.

(56) $\frac{2a^2}{x^2}+\frac{a}{3}$.

(57) $\frac{a-3x}{2ax}$.

(58) $\frac{ay+bx}{ay-bx}$.

(59) $\frac{1-x}{1+x}$.

(60) $\frac{3a}{2x}+\frac{4}{5}$.

---

(61) $2x-a$.

(62) $6+3a$.

(63) a^2x-a^3.

(64) $\frac{3}{4}-x$.

(65) $-\frac{3a}{10}$.

(66) -341.

---

(67) No. An Identity.

(68) $x=10$.

(69) $x=12$.

(70) $x=6$.

(71) $x=1$.

(72) $x=27$.

(73) $x=3$.

(74) $x=\frac{1}{2}$.

(75) $x=5$.

(76) $x=16$.

(77) $x=9$.

(78) $x=7$.

(79) $x=7$.

(80) $x=1$.

(81) $x=1$.

---

(82) $\left.\begin{array}{l} x=6, \\ y=8. \end{array}\right\}$

(83) $\left.\begin{array}{l} x=48, \\ y=56. \end{array}\right\}$

(84) $\left.\begin{array}{l} x=4, \\ y=3. \end{array}\right\}$

(85) $\left.\begin{array}{l} x=25, \\ y=5. \end{array}\right\}$

(86) $\left.\begin{array}{l} x=3, \\ y=4. \end{array}\right\}$

(87) $\left.\begin{array}{l} x=4, \\ y=7. \end{array}\right\}$

(88) $\left.\begin{array}{l} x=\frac{1}{2}, \\ y=\frac{1}{3}. \end{array}\right\}$

(89) $\left.\begin{array}{l} x=10, \\ y=16. \end{array}\right\}$

(90) 30.

(91) 14 half-crowns, and 49 shillings.

(92) $16\frac{1}{18}$, and $\frac{17}{18}$.

(93) 0·0165, and 0·0135.

(94) 15 miles, and 2 miles per hour.

(95) $16\frac{4}{11}$ min. past 3.

(96) $31\frac{1}{4}$ quarts of the best, and $18\frac{3}{4}$ of the other.

(97) 6 qrs. of wheat; 10 qrs. of barley.

(98) 4 sov., 59 sh., and 55 sixpences.

(99) 5 guineas; number of persons 15.

(100) 4.

MISCELLANEOUS EXERCISES.

SECOND SERIES.

(1) $-rx$.

(2) $(b-2)x^2$.

(3) $(3a-p-1)x^3$.

(4) No.

(5) Yes.

(6) $\frac{3a+1}{6}$.

(7) $a+b+c-d$.

(8) $2bx+2by$.

(9) $\frac{1}{2}x-\frac{a}{b}$.

(10) $\frac{5-x}{x-1}$.

(11) n.

(12) $5x$.

(13) x.

(14) $a^2+2ab+2b^2$.

(15) x^3-2x^2+x.

(16) $\frac{a}{b}$.

(17) $\frac{3a+2b}{3b-2a}$.

(18) $x^2+\frac{1}{x^2}+1$.

(19) $a^3+\frac{1}{a^3}+a+\frac{1}{a}$.

(20) $\frac{x^2}{4}+\frac{1}{x^2}-1$.

(21) x^6-a^6.

(22) $\sqrt{a}+\sqrt{b}$.

(23) $\frac{4x^2+9y^2}{4x^2-9y^2}$.

(24) $\frac{x(x+1)}{2}$.

---

(25) $x=4$.

(26) $x=12$.

(27) $x=8$.

(28) $x=3\frac{3}{4}$.

(29) $x=-\frac{3}{5}$.

(30) $x=\frac{2}{7}$.

(31) $x=-4\frac{1}{2}$.

(32) $x=6\frac{1}{2}$.

(33) $x=7\frac{13}{21}$.

(34) $x=\pm\frac{1}{2}\sqrt{5}$.

(35) $x=5$, or $5\frac{2}{5}$.

(36) $x=2$, or $-1\frac{3}{4}$.

(37) $x=2$, or $4\frac{6}{13}$.

(38) $x=2$, or $\frac{43}{87}$.

(39) $x=4$, or $3\frac{1}{2}$.

(40) $x=5$, or $6\frac{9}{10}$.

(41) $x=\frac{3a}{4}$, or $\frac{a}{2}$.

(42) $x=2$, or $-2\frac{7}{13}$.

(43) $\left.\begin{array}{l}x=8,\\y=2.\end{array}\right\}$

(44) $\left.\begin{array}{l}x=4,\\y=3.\end{array}\right\}$

(45) $\left.\begin{array}{l}x=131,\\y=31.\end{array}\right\}$

(46) $\left.\begin{array}{l}x=16,\\y=\ \ 4.\end{array}\right\}$

(47) $\left.\begin{array}{l}x=5,\\y=4\frac{1}{3}.\end{array}\right\}$

(48) $\left.\begin{array}{l}x=6,\\y=14.\end{array}\right\}$

(49) $\left.\begin{array}{l}x=7,\\y=12.\end{array}\right\}$

(50) $\left.\begin{array}{l}x=11,\\y=12.\end{array}\right\}$

(51) $\left.\begin{array}{l}x=\pm 3,\\y=3, \text{ or } \frac{1}{3}.\end{array}\right\}$

---

(52) 35 feet.
(53) £7500.
(54) 22, 18, 50.
(55) 22.
(56) *A* 5*s.*, *B* 3*s.*
(57) 320.
(58) 18, 38, 62.
(59) 216 apples, 96 oranges.

---

(60) 4 dozen.
(61) $\frac{n^2}{n+1}$, and $\frac{n}{n+1}$.
(62) 32*s.* and 14.
(63) 300.
(64) £8100, and 9.
(65) $7\frac{1}{2}$ miles, 100 yards.
(66) 355.
(67) 47.
(68) £70, £52, £40.
(69) 16.
(70) 52, and 48.
(71) $\frac{3}{7}$, $\frac{5}{9}$.
(72) 2, or 9, days.
(73) 15, 21, 24.
(74) From 3 to $2\frac{2}{5}$ per cent.
(75) $\left.\begin{array}{l}\text{112lbs. silver,}\\\text{36lbs. copper.}\end{array}\right\}$

---

(77) $x=\frac{a^2-b^2}{4a-b}$.
(79) 4, 10, 6.
(80) 27, 48.
(81) 6.
(82) 1 : 8.
(83) 3·3137 inches.
(84) 1 : 2.
(85) £5825. 8*s.* $5\frac{1}{3}d.$
(86) 123, 68, 16, 43.
(87) 8·24 in. nearly.
(88) 63 years, A.D. 1742.

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